

A Game Theoretic Model for the Formation of Navigable Small-World Networks — the Balance between Distance and Reciprocity *

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ABSTRACT

Kleinberg proposed a family of small-world networks to explain the navigability of large-scale real-world social networks. However, the underlying mechanism that drives real networks to be navigable is not yet well understood. In this paper, we present a game theoretic model for the formation of navigable small world networks. We model the network formation as a game called *Distance-Reciprocity Balanced (DRB)* game in which people seek for both high reciprocity and long-distance relationships. We show that the game has only two Nash equilibria: One is the navigable small-world network, and the other is the random network in which each node connects with each other node with equal probability, and any other network state can reach the navigable small world via a sequence of best-response moves of nodes. We further show that the navigable small world is very stable — (a) no collusion of any size would benefit from deviating from it; and (b) after an arbitrary deviations of a large random set of nodes, the network would return to the navigable small world as soon as every node takes one best-response step. In contrast, for the random network, any random deviation of one node or any collusion of two neighboring nodes will bring the network out of the random-network equilibrium, and a small group collusion or random perturbations is guaranteed to move the network to the navigable network as soon as every node takes one best-response step. Moreover, we show that navigable small world has much better social welfare than the random network, and provide the price-of-anarchy and price-of-stability results of the game. Our empirical evaluation further demonstrates that the system always converges to the navigable network even when limited or no information about other players' strategies is available, and the DRB game simulated on the real-world networks leads to navigability characteristic that is very close to that of the real networks, even though the real-world network has non-uniform population distribution different from the Kleinberg's small-world model. Our theoretical and empir-

ical analyses provide important new insight on the connection between distance, reciprocity and navigability in social networks.

Categories and Subject Descriptors

G.2.2 [Discrete Mathematics]: Graph Theory—*Network problems*

General Terms

Theory

Keywords

small-world network, game theory, navigability, reciprocity

1. INTRODUCTION

In 1967, Milgram published his work on the now famous small-world experiment [31]: he asked test subjects to forward a letter to their friends in order for the letter to reach a person not known to the initiator of the letter. He found that on average it took only six hops to connect two people in U.S., which is often attributed as the source of the popular term *six-degree of separation*. This seminal work inspired numerous researches on the small-world phenomenon and small-world models, which last till the present day of information age.

In [37] Watts and Strogatz investigated a number of real-world networks such as film actor networks and power grids, and showed that many networks have both low diameter and high clustering (meaning two neighbors of a node are likely to be neighbors of each other), which is different from randomly wired networks. They thus proposed a small-world model in which nodes are first placed on a ring or a grid with local connections, and then some connections are randomly rewired to connect to long-range contacts in the network. The local and long-range connections can also be viewed as strong ties and weak ties respectively in social relationships originally proposed by Granovetter [14, 13].

Kleinberg notices an important discrepancy between the small-world model of Watts and Strogatz and the original Milgram experiment: the latter shows not only that the average distance between nodes in the network are small, but also that nodes can *efficiently navigate* in the network with *only local information*. To address this issue, Kleinberg adjusted Watts-Strogatz model so that the long-range connections are selected not uniformly at random among all nodes

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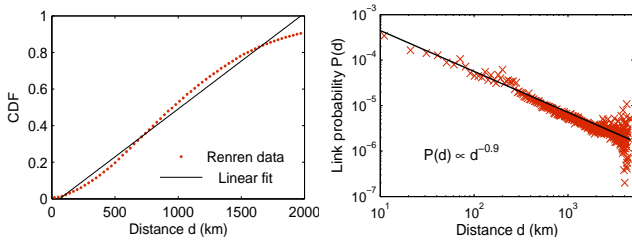


Figure 1: The fraction of Friendship nodes within distance d in Renren.

but inversely proportional to a power of the grid distance between the two end points of the connection [22].

More specifically, Kleinberg modeled a social network as composed of n^k nodes on a k -dimensional grid, with each node having local contacts to other nodes in its immediate geographic neighborhood. Each node u also establishes a number of long-range contacts, and a long-range link from u to v is established with probability proportional to $d_M(u, v)^{-r}$, where $d_M(u, v)$ is the grid distance between u and v , and $r \geq 0$ is the model parameter indicating how likely nodes prefer to connect to remote nodes, which we call *connection preference* in the paper. Watts-Strogatz model corresponds to the case of $r = 0$, and as r increases, nodes are more likely to connect to other nodes in their vicinity. Kleinberg modeled Milgram’s experiment as decentralized greedy routing in such networks, in which each node only forwards messages to one of its neighbors with coordinate closest to the target node. He showed that when $r = k$, greedy routing can be done efficiently in $O(\log^2 n)$ time in expectation, but for any $r \neq k$, it requires $\Omega(n^c)$ time for some constant c depending on r , exponentially worse than the case of $r = k$. Therefore, the small world at the critical value of $r = k$ is meant to model the real-world navigable network validated by Milgram and others’ experiments, and we call it the *navigable small-world network*.

After Kleinberg’s theoretical analysis, a number of empirical studies have been conducted to verify if real networks indeed have connection preference close to the critical value that allows efficient greedy routing [29, 1, 7, 10, 35]. Since real network population is not evenly distributed geographically as in the Kleinberg’s model, Liben-Nowell et al. [29] proposed to use the *fractional dimension* α , defined as the best value to fit $|\{w : d_M(u, w) \leq d_M(u, v)\}| = c \cdot d_M(u, v)^\alpha$, averaged over all u and v . They showed that when the connection preference $r = \alpha$, the network is navigable. They then studied a network of 495,836 LiveJournal users in the continental United States who list their hometowns, and find that $\alpha \approx 0.8$ while $r = 1.2$, reasonably close to α . We apply the same approach to a ten million node Renren network [20, 38], one of the largest online social networks in China. We map the hometown listed in users’ profiles to (longitude, latitude) coordinates. The resolution of our geographic data is limited to the level of towns and cities and thus we cannot get the exact distance of nodes within 10km. We found that $\alpha \approx 1$ (Figure 1) and $r \approx 0.9$ (Figure 2) in Renren network. Other studies [1, 7, 10, 35] also reported connection preference r to be close to 1 in other online social networks (including Gowalla, Brightkite and Facebook). Even though they did not report the fractional dimension, from both the

LiveJournal data in [29] and our Renren data, it is reasonable to believe that the fractional dimension is also close to 1. Therefore, empirical evidences all suggest that the real-world social networks indeed have connection preference close to the critical value and the network is navigable.

A natural question to ask next is how navigable networks naturally emerge? What are the forces that make the connection preference become close to the critical value? As Kleinberg pointed out in his survey paper [23] when talking about the above striking coincidence between theoretical prediction and empirical observation, “it suggests that there may be deeper phenomena yet to be discovered here”. There are several studies trying to explain the emergence of navigable small-world networks [30, 17, 8, 34, 5], mostly by modeling certain underlying node or link dynamics (see related work for more details).

In this paper, we tackle the problem in a novel way using game-theoretic approach, which is reasonable in modeling individual behaviors in social networks without central coordination. One key insight we have is that connection preference r is not a global preference but individual’s own preference — some prefer to connect to more faraway nodes while others prefer to connect to nearby nodes. Therefore, we establish *small-world formation games* where individual node u ’s strategy is its own connection preference r_u (Section 3). This game formulation is different from most existing network formation games where individuals’ strategies are creating actual links in the network (c.f. [36]). It allows us to directly explore the entire parameter space of connection preferences and answer the question on why nodes end up choosing a particular parameter setting leading to the navigable small world.

In terms of payoff functions, we first consider minimizing greedy routing distance to other nodes as the payoff, since it directly corresponds to the goal of navigable networks. However, Gulyás et al. [16] prove that with this payoff the navigable networks cannot emerge as an equilibrium for the one-dimensional case. Our empirical analysis also indicates that nodes will converge to random networks ($r_u = 0, \forall u$) rather than navigable networks for higher dimensions. Our empirical analysis further shows that if we adjust the payoff with a cost proportional to the grid distance of remote connections, the equilibria are sensitive to the cost factor.

The above unsuccessful attempt suggests that besides the goal of shortening distance to remote nodes, some other natural objective may be in play. Reciprocity is regarded as a basic mechanism that creates stable social relationships in a person’s life [12]. A number of prior works [19, 29, 32] also suggest that people seek reciprocal relationships in online social networks. Therefore, we propose a payoff function that is the product of average distance of nodes to their long-range contacts and the probability of forming reciprocal relationship with long-range contacts. We call this game distance-reciprocity balanced (DRB) game. In practice, increasing relationship distance captures that individuals attempt to create social bridges by linking to “distant people”, which can help them search for and obtain new resources. Meanwhile, increasing reciprocity captures that individuals look at social bonds by linking to “people like them”, which could help them preserve or maintain resources. Therefore, the DRB game is natural since it captures sources of bridging and bonding social capital in building social integration and solidarity [9].

Even though the payoff function for the DRB game is very simple, our analysis demonstrates that it is extremely effective in producing navigable small-world networks as the equilibrium structure. In theoretical analysis (Section 4), we first show that navigable small world ($r_u = k, \forall u$) and random small world ($r_u = 0, \forall u$) are the only two Nash equilibria of the DRB game, and for any strategy profile that is not the random network, it can always reach the navigable small world through a cascade of nearby nodes adopting strategy k in a best-response dynamic.

In terms of the stability of NE, we prove that the navigable small world is a strong Nash equilibrium, which means that it tolerates collusion of any size trying to gain better payoff. Moreover, it also tolerates arbitrary deviations (without the objective of increasing anyone's payoff) of large groups of random deviators, since the system is guaranteed to return back to the navigable NE as soon as every node takes one best-response step. In contrast, random small world can be moved away from its equilibrium state by either a random perturbation of one node or a collusion of two nearby nodes, and when a small random set of nodes perturb to different strategies, we prove that the system is guaranteed to converge to the navigable small world as soon as every node takes one best-response step. Our theoretical analysis provides strong support that navigable small-world network is the unique and stable equilibrium that would naturally emerge in the DRB game.

We further examine the global function of social welfare (i.e., the total payoff of all nodes), as what is the optimal social welfare, and how selfish behavior of users affect the social welfare. Interestingly, we find that the global optimum can be reached by a fraction of nodes sacrificing their distance payoff to focus on reciprocity (by selecting a strategy greater than k) so that their neighbors could select strategy k to reach a high balanced payoff of both distance and reciprocity. This situation reminds us social relationships generated by different social status (e.g. employee-employer relationship) or by tight bound with mutual understanding and support (such as marriage relationship). Next we compare the social welfare of navigable and random small-world networks with the global optimum through the standard price of anarchy (PoA) and price of stability (PoS) metrics, which is the ratio of social welfare between the global optimum and the worst (or the best) Nash equilibrium, respectively. We show that $PoA = \Theta\left(\frac{n^k}{\ln^2 n}\right)$ and $PoS = \Theta(\ln n)$ with navigable network having the better social welfare, which means that the navigable small world has polynomially better social welfare than the random small world, while is only logarithmically worse than the global optimum.

To complement our theoretical analysis, we conduct empirical evaluations to cover more realistic game scenarios not covered by our theoretical analysis (Section 6). We first test random perturbation cases and show that arbitrary initial profiles always converge to the navigable equilibrium in a few steps, while a very small random perturbation (less than theoretical prediction) of the random small world causes it to quickly converge back to the navigable equilibrium. Next, we simulate more realistic scenarios where nodes have limited or no information about other nodes' strategies. We show that if they only learn their friends' strategies (with some noise), the system still converges close to the navigable equilibrium in a small number of steps. Further, even when

the node has no information about other players' strategies and can only use its obtained payoff as feedback to search for the best strategy, the system still moves close to the navigable equilibrium within a few hundred steps (in the 100×100 grid).

Finally we simulate the DRB game on Renren and LiveJournal networks, which have non-uniform population distributions different from Kleinberg's grid-based small-world model. Our simulation results show that in both networks, the game quickly converges to an equilibrium where connection preferences of users are close to the empirical ones at both global and city levels. We further demonstrate that individual connection preference is negatively correlated with population density nearby the individual — people living in large cities are more likely to adopt a relatively small value of strategy r . We argue that our DRB game provides a reasonable explanation for this phenomenon: people in densely populated cities are easier to achieve reciprocity and thus they would focus more on connection distance (meaning choosing a smaller connection preference value r). These empirical results further demonstrate the robustness of the navigable small world in the DRB game.

In summary, our contributions are the following: (a) we propose the small-world formation game and design a balanced distance-reciprocity payoff function to explain the navigability of real social networks; (b) we conduct comprehensive theoretical and empirical analysis to demonstrate that navigable small world is the unique robust equilibrium that would naturally emerge from the game under both random perturbation and strategic collusions; and (c) our game reveals a new insight between distance, reciprocity and navigability in social networks, which may help future research in uncovering deeper phenomena in navigable social networks. To our best knowledge, this is the first game theoretic study on the emergence of navigable small-world networks, and the first study that linking relationship reciprocity with network navigability.

Additional related work. We provide additional details of prior works on explaining the emergence of navigable small-world networks, and other related studies not covered in the introduction.

Some studies try to explain navigability by assuming that nodes form links to optimize for a particular property. Mathias et al. [30] assume that users try to make trade-off between wiring and connectivity. Hu et al. [17] assume that people try to maximize the entropy under a constraint on the total distances of their long-range contacts. These works rely on simulations to study the network dynamics. Moreover, the navigability of a network is sensitive to the weight of wiring cost or the distance constraint, and it is unlikely that navigable networks as defined by Kleinberg [22] would naturally emerge.

Another type of works propose node/link dynamics that converge to navigable small-world networks. Clauset and Moore [8] propose a rewiring dynamic modeling a Web surfer such that if the surfer does not find what she wants in a few steps of greedy search, she would rewire her long-range contact to the current end node of the greedy search. They use simulations to demonstrate that a network close to Kleinberg's navigable small world will emerge after long enough rewiring rounds. Sandberg and Clarke [34] propose another rewiring dynamic where with an independent probability of p each node on a greedy search path would rewire

their long-range contacts to the search target, and provide a partial analysis and simulations showing that the dynamic converges to a network close to the navigable small world. Chaintreau et al. [5] use a move-and-forget mobility model, in which a token starting from each node conducts a random walk (move) and may also go back to the starting point (forget), and use the distribution of the token on the grid as the distribution of the long-range contacts of the starting node. They provide theoretical analysis showing that the move-and-forget model with a particular harmonic forget function converges close to the navigable small world. However, it is unclear if the harmonic forget function used is natural in practice and what is the effect of other forget functions.

The approach taken by these studies can be viewed as orthogonal and complementary to our approach: they aim at using natural dynamics (rewiring or mobility dynamics) to explain navigable small world, while we focus on directly exploring the entire parameter space of connection preferences of nodes and use game theoretic approach to show, both theoretically and empirically, that the nodes would naturally choose their connection preferences to form the navigable small world. Moreover, all the prior studies only show that they converge approximately to the navigable small world, while in our game the navigable small world is precisely the only robust equilibrium. Finally, none of these works introduce reciprocity in their model and we are the first to link reciprocity with navigability of the small world.

Some studies use hyperbolic metric spaces or graphs to try to explain navigability in small-world networks (e.g. [4, 33, 24, 25, 6, 15]). However, they do not explain why connection preferences in real networks are around the critical value and how navigable networks naturally emerge. In particular, Chen et al. [6] show that the navigable small world in Kleinberg's model does not have good hyperbolicity. Most recently, Gulyás et al. [15] propose a game where each player tries to minimize the number of links in order to be able to greedily route to all other nodes. The equilibrium of the game is a scale-free network whose degree distribution follows a power law. However, this game is not intended and does not explain the emergence of navigable small-world network validated by Milgram and others' experiments, where greedy routing can be done efficiently in $O(\log^2 n)$ time in expectation, and relationship reciprocity is not included in any aspect of the game.

A number of network dynamics are proposed to address general network evolution, but they do not address network navigability in particular. For example, models in [3, 28, 27, 11] leverage preferential attachment or triangle closure mechanisms to capture power-law degree or high clustering coefficient, and other models [2, 21] capture spatial effects using a gravity model, balancing the effect of spatial distance with other node properties (e.g., node degree).

2. PRELIMINARIES

In this section, we present the Kleinberg's small-world model and some basic concepts of a noncooperative game.

2.1 Kleinberg's Small-World Model

Let $V = \{(i, j) : i, j \in [n] = \{1, 2, \dots, n\}\}$ be the set of n^2 nodes forming an $n \times n$ grid. For convenience, we consider the grid with wrap-around edges connecting the nodes on the two opposite sides, making it a torus. For any two nodes $u = (i_u, j_u)$ and $v = (i_v, j_v)$ on this wrap-around

grid, the *grid distance* or *Manhattan distance* between u and v is defined as $d_M(u, v) = \min\{|i_v - i_u|, n - |i_v - i_u|\} + \min\{|j_v - j_u|, n - |j_v - j_u|\}$.

The model has two universal constants $p, q \geq 1$, such that (a) each node has undirected edges connecting to all other nodes within lattice distance p , called its *local contacts*, and (b) each node has q random directed edges connecting to possibly faraway nodes in the grid called its *long-range contacts*, drawn from the following distribution. Each node u has a *connection preference* parameter $r_u \geq 0$, such that the i -th long-range edge from u has endpoint v with probability proportional to $1/d_M(u, v)^{r_u}$, that is, with probability $p_u = d_M(u, v)^{-r_u}/c(r_u)$, where $c(r_u) = \sum_{v \neq u} d_M(u, v)^{-r_u}$ is the normalization constant. Let \mathbf{r} be the vector of r_u values on all nodes. We use $\mathbf{r} \equiv s$ to denote $r_u = s, \forall u \in V$.

Greedy routing on the small-world network from a source node u to a target node v is a decentralized algorithm starting at node u , and at each step if routing reaches a node w , then w selects one node from its local and long-range contacts that is closest to v in grid distance as the next step in the routing path, until it reaches v . In [22], Kleinberg shows that when $\mathbf{r} \equiv 2$, the expected number of greedy routing steps (called *delivery time*) is $O(\log^2 n)$, but when $\mathbf{r} \equiv s \neq 2$, it is $\Omega(n^c)$ for some constant c related to s .

The above model can be easily extended to k dimensional grid (with wraparound) for any $k = 1, 2, 3, \dots$, where each long range contact is still established with probability proportional to $1/d_M(u, v)^{r_u}$. It is shown that $\mathbf{r} \equiv k$ is the critical value allowing efficient greedy routing. Henceforth, we call Kleinberg's small world with $\mathbf{r} \equiv k$ the *navigable small world*. Another special network is $\mathbf{r} \equiv 0$, in which every node's long-range contacts are selected among all nodes uniformly at random, and we refer it as the *random small world*. We use $K(n, k, p, q, \mathbf{r})$ to refer to the class of Kleinberg random graphs with parameters n, k, p, q , and \mathbf{r} .

2.2 Game and Solution Concepts

A game is described by a system of players, strategies and payoffs. We denote a game by $\Gamma = (S_u, \pi_u)_{u \in V}$, where V represents a finite set of players, S_u is the set of strategies of player u , and $\pi_u : \mathcal{S} \rightarrow \mathbb{R}$ is the payoff function of node u , with $\mathcal{S} = S_1 \times S_2 \times \dots \times S_n$. An element $\mathbf{s} = (s_1, s_2, \dots, s_n) \in \mathcal{S}$ is called a *strategy profile*.

Let $\mathcal{C} = 2^V \setminus \emptyset$ denote the set of all coalitions. For each coalition $C \in \mathcal{C}$, let $-C = V \setminus C$, and if $C = \{u\}$, we denote $-C$ by $-u$. We also denote by \mathcal{S}_C the set of strategies of players in coalition C , and \mathbf{s}_C the partial strategy profile of \mathbf{s} for nodes in C .

DEFINITION 1 (BEST RESPONSE). *Player u 's strategy $s_u^* \in S_u$ is a best response to the strategy profile $\mathbf{s}_{-u} \in \mathcal{S}_{-u}$ if*

$$\pi_u(s_u^*, \mathbf{s}_{-u}) \geq \pi_u(s_u, \mathbf{s}_{-u}), \forall s_u \in S_u \setminus \{s_u^*\},$$

Moreover, if " \geq " above is actually " $>$ " for all $s_u \neq s_u^$, then s_u^* is the unique best response to \mathbf{s}_{-u} .*

Nash equilibrium (NE) for a strategic game is a strategy profile such that each player's strategy is a best response to the other players' strategies.

DEFINITION 2 (NASH EQUILIBRIUM). *A strategy profile $\mathbf{s}^* \in \mathcal{S}$ is a Nash equilibrium if for every player $u \in V$, s_u^* is a best response to \mathbf{s}_{-u}^* ; \mathbf{s}^* is a strict Nash equilibrium if for every player $u \in V$, s_u^* is the unique best response to \mathbf{s}_{-u}^* .*

For a game, we often study its *best response dynamic* to investigate its properties of convergence to Nash equilibria. Best response dynamic is typically specified in terms of *asynchronous steps*: in each asynchronous step, *one* player moves from its current strategy to its best response to the current strategy profile, and thus the entire strategy profile moves one step accordingly. To facilitate the study of convergence speed, we also look into *synchronous steps* for the best response dynamic: in each synchronous step, *every* player moves from its current strategy to its best response to the current strategy profile, and collectively we count this as one synchronous step.

While in an NE no player can improve its payoff by unilateral deviation, some of the players may benefit (sometimes substantially) from forming alliances/coalitions with other players. The *strong Nash equilibrium (SNE)* is a strategy profile for which no coalition of players has a joint deviation that improves the payoff of each member of the coalition.

DEFINITION 3 (STRONG NASH EQUILIBRIUM). For a real number $f \in (0, 1]$, a strategy profile $\mathbf{s}^* \in \mathcal{S}$ is an t -strong Nash equilibrium if for all $C \in \mathcal{C}$ with $|C| \leq t$, there does not exist any $\mathbf{s}_C \in \mathcal{S}_C$ such that

$$\forall u \in C, \pi_u(\mathbf{s}_C, \mathbf{s}_{-C}^*) \geq \pi_u(\mathbf{s}^*), \exists u \in C, \pi_u(\mathbf{s}_C, \mathbf{s}_{-C}^*) > \pi_u(\mathbf{s}^*).$$

When $t = |V|$, we simply call \mathbf{s}^* the strong Nash equilibrium.

Note that 1-SNE falls back to NE, while $|V|$ -SNE or simply SNE means *Pareto-optimal*, which requires that no player can improve her payoff without decreasing the payoff of someone else. Therefore, SNE is a very strong equilibrium concept allowing collusions of any size.

For a game, we often use social welfare (defined below) as a benchmark to compare the social efficiency of its Nash equilibria against the global optimal solution.

DEFINITION 4 (SOCIAL WELFARE). The social welfare of a strategy profile $\mathbf{s} \in \mathcal{S}$ of a game is the sum of the payoffs of all players: $SW(\mathbf{s}) = \sum_{u \in V} \pi_u(\mathbf{s}_u, \mathbf{s}_{-u})$.

Let \mathcal{S}^* be the set of the Nash equilibria of the game, and we define the following measures:

DEFINITION 5 (PRICE OF ANARCHY). Price of anarchy (PoA) is the ratio of the social welfare from the global optimum to the worst Nash equilibrium, which is:

$$PoA = \frac{\max_{\mathbf{s} \in \mathcal{S}} SW(\mathbf{s})}{\min_{\mathbf{s}^* \in \mathcal{S}^*} SW(\mathbf{s}^*)}.$$

DEFINITION 6 (PRICE OF STABILITY). Price of stability (PoS) is the ratio of the social welfare from the global optimum to the best Nash equilibrium, which is:

$$PoS = \frac{\max_{\mathbf{s} \in \mathcal{S}} SW(\mathbf{s})}{\max_{\mathbf{s}^* \in \mathcal{S}^*} SW(\mathbf{s}^*)}.$$

3. SMALL-WORLD FORMATION GAMES

Connection preference r_u in Kleinberg's model reflects u 's intention in establishing long-range contacts: When $r_u = 0$, u chooses its long-range contacts uniformly among all nodes in the grid; as r_u increases, the long-range contacts of u become increasingly clustered in its vicinity on the grid. Our insight is to treat connection preference as node's strategy in a game setting and study the game behavior.

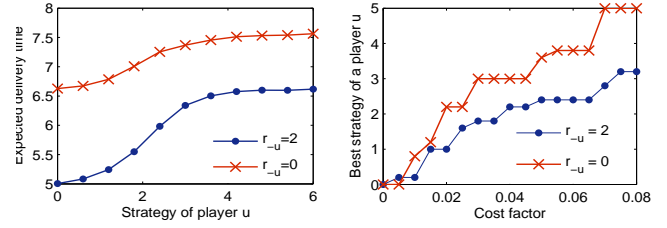


Figure 3: The expected delivery time for a player u with different strategy. **Figure 4: The best response of a player u given u with different cost factor.**

More specifically, we model this via a non-cooperative game among nodes in the network. First, we assume that each r_u is taken from a discrete set $\Sigma = \{0, \gamma, 2\gamma, 3\gamma, \dots\}$, where γ represents the granularity of connection preference and is in the form of $1/g$ for some positive integer $g \geq 2$. Using discrete strategy set avoids nuances in continuous strategy space and is also reasonable in practice since people are unlikely to make infinitesimal changes. Next, we model the small-world network formation as a game $\Gamma = (\Sigma, \pi_u)_{u \in V}$, where V is the set of nodes in the grid, connection preference $r_u \in \Sigma$ is the strategy of a player u , and π_u is the payoff function of u . Our objective is to study intuitively appealing payoff functions π_u and find one that would allow the navigable small-world network to emerge.

3.1 Routing-based Payoff

As navigable small world achieves best greedy routing efficiency, it is natural to consider the payoff function as the expected delivery time to the target in greedy routing. Given the strategy profile $\mathbf{r} \in \mathcal{S}$, let $t_{uv}(r_u, \mathbf{r}_{-u})$ be the expected delivery time from source u to target v via greedy routing. The payoff function is given by:

$$\pi_u(r_u, \mathbf{r}_{-u}) = - \sum_{v \neq u} t_{uv}(r_u, \mathbf{r}_{-u}). \quad (1)$$

We take a negation on the sum of expected delivery time because nodes prefer shorter delivery time.

Although the above payoff function is intuitive and simple, it has some serious issues. Prior work [16] has already proved that, with the length of greedy paths as the payoff, player u 's best response is to link uniformly (i.e., $r_u = 0$) for the one-dimensional case. For higher dimensions, Figure 3 shows the expected delivery time for a single node u at a 100×100 grids, where each node generates $q = 10$ links. We see that when other nodes fixed their strategy (e.g., $\mathbf{r}_{-u} \equiv 2$), the best strategy of a single node u is 0. More tests on different initial conditions reach the same result that the system will converge to the random small-world networks. The intuitive reason is that to reach other nodes quickly, it is better for a node to evenly spread its long-range contacts in the network. This is inconsistent with empirical evidence that real-world networks are navigable ones ([29, 7, 1, 10, 26, 18]).

In practice, creating and maintaining long-range links have higher costs, so one may adapt the above payoff function by adding the grid distances of long-range contacts as a cost term in the payoff function:

$$\pi_u(r_u, \mathbf{r}_{-u}) = - \sum_{v \neq u} t_{uv}(r_u, \mathbf{r}_{-u}) - \lambda \sum_{v \neq u} p_u(v, r_u) d_M(u, v), \quad (2)$$

where λ is a factor controlling the long range cost. A larger λ means users are more concerned with distance costs. Figure 4 shows that the best strategy of a user u is significantly influenced by the cost factor. Similar result is also shown in [16]. Thus, it is unclear if the navigable small-world network can naturally emerge from this type of games.

3.2 Distance-Reciprocity Balanced Payoff

The previous section demonstrates that seeking short routing distance alone cannot explain the emergence of navigable small world, and thus people in the social network must have some other objective to achieve. Reciprocity is regarded as a basic mechanism that creates stable social relationships in the real world [12]. Several empirical studies [19, 29, 32] also show that high reciprocity is also a typical feature present in real small-world networks (such as Flickr, YouTube, LiveJournal, Orkut and Twitter).

Therefore, we consider the payoff of a user u as the following balanced objective between distance and reciprocity:

$$\pi_u(r_u, \mathbf{r}_{-u}) = \left(\sum_{v \neq u} p_u(v, r_u) d_M(u, v) \right) \times \left(\sum_{v \neq u} p_u(v, r_u) p_v(u, r_v) \right), \quad (3)$$

where $\sum_{v \neq u} p_u(v, r_u) d_M(u, v)$ is the mean grid distance of u 's long-range contacts, and $\sum_{v \neq u} p_u(v, r_u) p_v(u, r_v)$ is the mean probability for u to form bi-directional links with its long-range contacts, i.e., reciprocity. We refer the small-world formation game with payoff function in Eq.(3) the *Distance-Reciprocity Balanced (DRB) game*.

The payoff function in Eq.(3) reflects two natural objectives users in a social network want to achieve: first, they want to connect to remote nodes, which may give them diverse information as in the famous "the strength of weak tie argument" by Granovetter [14]; second, they want to establish reciprocal relationship which are more stable in long term. However, these two objectives can be in conflict for a node u when others prefer linking in their vicinity (i.e., other nodes v choosing positive exponent r_v). In this case, faraway long-range contacts are less likely to create reciprocal links. Therefore, node u should obtain the maximum payoff when it achieves a balance between the two objectives. We use the simple product of distance and reciprocity objectives to model this balancing behavior.

We remark that the reciprocity term $\sum_{v \neq u} p_u(v, r_u) p_v(u, r_v)$ does not consider reciprocity formed by fixed local contacts. Effectively, we disregard local contacts and treat $p = 0$ in the small world setting $K(n, k, p, q, \mathbf{r})$. This treatment makes our analysis more streamlined and only focus on long-range contacts, and it also makes intuitive sense: the local contacts are passively given based on geographic location, while long-range contacts are actively established by nodes based on their connection preference, and thus reciprocity based on long-range contacts could make more sense. For example, your neighbors in the same apartment building are your local contacts by physical location, but it does not mean that they are your friends, and you still need to intentionally establish friendship (based on your preference) among your neighbors, and thus reciprocity only by physical location

does not mean much but reciprocity based on actively established relationship does mean a lot for an individual.

4. PROPERTIES OF THE DRB GAME

In this section, we conduct theoretical analysis to discover the properties of the DRB game. When player u 's strategy r_u is the unique best response to a strategy profile \mathbf{r}_{-u} , we denote this unique best response as $B_u(\mathbf{r}_{-u})$.

4.1 Equilibrium Existence and Best Response Dynamic

We first show that navigable small-world network is a Nash Equilibrium of the DRB game.

To do so, we focus on a local region centering around a node w preferring local connection, and we have the following important lemma.

LEMMA 1. *In the k -dimensional DRB game, for any constant δ , there exists $n_0 \in \mathbb{N}$ (may depend on δ), for any $n \geq n_0$, for any non-zero strategy profile $\mathbf{r} \neq 0$, for any node w satisfying $r_w \geq k$ or $r_w = \max_{v \in V} r_v$, for any u within δ grid distance of w (i.e. $d_M(u, w) \leq \delta$), u has the unique best response of $r_u = k$.*

PROOF (SKETCH). The intuition is as follows. When a node w satisfying $r_w \geq k$ or $r_w = \max_{v \in V} r_v$, it prefers its long-range contacts to be in its vicinity. For a nearby node u with $d_M(u, w) \leq \delta$, the case of $r_u = k$ provides the best balance between good grid distance to long-range contacts and high reciprocity (even just counting the reciprocity received from w). In other cases, the node u obtains either too low reciprocity or too short average grid distance to long-range contacts. The detailed proof is included in Appendix B. \square

The above lemma shows that given a non-zero profile, we can find a local region where the best response of every node is k . This lemma is instrumental to several analytical results, including the following theorem.

THEOREM 1. *For the DRB game in a k -dimensional grid, the following is true for sufficiently large n :¹ For every node $u \in V$, every strategy profile \mathbf{r} , and every $s \in \Sigma$, if $\mathbf{r}_{-u} \equiv s$, then u has a unique best response to $\mathbf{r}_{-u} \equiv s$:*

$$B_u(\mathbf{r}_{-u} \equiv s) = \begin{cases} k & \text{if } s > 0, \\ 0 & \text{if } s = 0. \end{cases}$$

PROOF (SKETCH). For the case of $s > 0$, given the strategy profile of $\mathbf{r}_{-u} \equiv s$, for every node u , each of its nearest neighbor w (i.e., $d_M(u, w) = 1$) satisfies $r_w = \max_{v \in V} r_v$. Thus by Lemma 1, node u 's unique best response to $\mathbf{r}_{-u} \equiv s$ is $r_u = k$.

When $s = 0$, all others nodes link uniformly. In this case, the reciprocity for node u becomes a constant independent of its strategy r_u . Thus, r_u should be selected to maximize average distance of u 's long-range contacts, which leads to $r_u = 0$. The detailed proof of this case is included in Appendix C. \square

¹Technically, a statement being true for sufficiently large n means that there exists a constant $n_0 \in \mathbb{N}$ that may only depend on model constants such as k and γ , such that for all $n \geq n_0$ the statement is true in the grid with parameter n .

Theorem 1 shows that when all other nodes use the same nonzero strategy s , it is strictly better for u to use strategy k ; when all other nodes uniformly use the 0 strategy, it is strictly better for u to also use 0 strategy. When setting $s = k$ and $s = 0$, we have:

COROLLARY 2. *For the DRB game in the k -dimensional grid, the navigable small-world network ($\mathbf{r} \equiv k$) and the random small-world network ($\mathbf{r} \equiv 0$) are the two strict Nash equilibria for sufficiently large n , and there are no other uniform Nash equilibria.*

We next examine if there exists any non-uniform equilibrium.

THEOREM 3. *In the k -dimensional DRB game, there is no non-uniform Nash equilibrium for sufficiently large n .*

PROOF. Given any non-uniform strategy profile \mathbf{r} , let $V_{\geq k} = \{v | r_v \geq k\}$. If $V_{\geq k} \neq \emptyset$, we can find a pair of grid neighbors (u, w) with $r_u \neq k$ and $r_w \geq k$. If $V_{\geq k} = \emptyset$, we can find a pair of grid neighbors (u, w) with $r_u \neq k$ and $r_w = \max_{v \in V} r_v$. In either case, we know the node u could obtain better payoff by unilaterally deviate to the strategy $r_u = k$ by Theorem 1. Therefore, non-uniform strategy profile \mathbf{r} is not a Nash equilibrium. \square

Combining the above theorem with Corollary 2, we see that DRB game has *only two* Nash equilibria $\mathbf{r} \equiv k$ and $\mathbf{r} \equiv 0$, corresponding to the navigable and random small-world networks, respectively.

Next, we show that for any non-zero profile, we can find a node that triggers a cascade of adopting strategy k from neighbors to neighbors of neighbors, and so on, ultimately leading to the navigable small world equilibrium.

THEOREM 4. *In the k -dimensional DRB game, for sufficiently large n , the navigable small-world equilibrium $\mathbf{r} \equiv k$ is reachable via best response dynamic from any non-zero strategy profile $\mathbf{r} \neq 0$. Moreover, if all nodes move synchronously in the best response dynamic, then it takes at most $k \lfloor n/2 \rfloor$ synchronous steps for any non-zero strategy profile to converge to the navigable small-world equilibrium $\mathbf{r} \equiv k$.*

PROOF. Let $V_w(j) = \{v | d_M(v, w) \leq j\}$. Given a non-zero profile \mathbf{r} , we can find a node w satisfying $r_w \geq k$ or $r_w = \max_{v \in V} r_v$. Given a constant δ ($\delta \geq 2$), Lemma 1 implies that for sufficiently large n , for every $u \in V_w(\delta)$, in one asynchronous step u will set $r_u = k$. Then consider u 's neighbors $V_u(\delta)$, in one asynchronous step each of them will also set their strategy to k . Following this cascade it is clear that there exists a step sequence such that the non-zero profile \mathbf{r} will reach the navigable small world $\mathbf{r} \equiv k$.

We now consider that all nodes move synchronously. Again we first find a node w satisfying $r_w \geq k$ or $r_w = \max_{v \in V} r_v$. By Lemma 1 all nodes in $V_w(\delta/2)$ ($\delta/2 \geq 1$) move to strategy k in the first synchronous step. Consider the second synchronous step. Even though we are not sure if node w adopts strategy k in the first synchronous step, we know that w adopts k in the second synchronous step since w has neighbors adopting k after the first synchronous step. Moreover, for all nodes in $V_w(\delta/2)$, their mutual grid distance is at most δ , and thus Lemma 1 applies to these nodes in the second synchronous step and they all stay at

strategy k . Finally for their grid neighbors within grid distance $\delta/2$, essentially nodes in $V_w(\delta) \setminus V_w(\delta/2)$, they will also adopt strategy k in the second synchronous step. Repeating the above procedure, all nodes that have adopted k will keep k while their grid neighbors will also adopt k . Since the longest grid distance among nodes in $K(n, k, p, q, \mathbf{r})$ is $k \lfloor n/2 \rfloor$, after at most $k \lfloor n/2 \rfloor$ synchronous steps, all nodes adopt k . \square

4.2 Equilibrium Stability in DRB Game

In this section, we show the important results that the navigable small-world network is stable in terms of tolerating both collusions of any group of players and arbitrary deviations of random players' strategies, while random equilibrium do not tolerate either collusions or random perturbations of a small group of players.

We first show that the navigable small-world network tolerates collusion of players of any size.

THEOREM 5. *For the DRB game in the k -dimensional grid, the navigable small-world network ($\mathbf{r} \equiv k$) is a strong Nash equilibrium for sufficiently large n .*

PROOF (SKETCH). We prove a slightly stronger result — any node u in any strategy profile \mathbf{r} with $r_u \neq k$ has strictly worse payoff than its payoff in the navigable small world. Intuitively, when u deviates to $0 \leq r_u < k$, its loss on reciprocity would outweigh its gain on link distance; when u deviates to $r_u > k$, its loss on link distance is too much to compensate any possible gain on reciprocity. The detailed proof is in Appendix D. \square

The above theorem shows that the navigable small-world equilibrium is not only immune to unilateral deviations, but also to deviations by coalitions, and in particular it is Pareto-optimal, such that no player can improve her payoff without decreasing the payoff of someone else.

After showing that the navigable small-world is robust to collusion, we now show that random small world equilibrium is not stable even under the collusion of a pair of nodes.

THEOREM 6. *For the DRB game in a k -dimensional grid, the random small-world NE $\mathbf{r} \equiv 0$ is not a 2-strong Nash equilibrium for sufficiently large n .*

PROOF (SKETCH). If a pair of grid neighbors collude to deviate their strategies to k , they could gain much benefit in terms of reciprocity, as compared with the loss of relationship distance. As a result, they would both get better payoff than their payoff in $\mathbf{r} \equiv 0$. The detailed proof is in Appendix E. \square

Next, we would like to see if the navigable equilibrium can also tolerate deviations of players, even if the deviations could be arbitrary and there is no guarantee that deviated players are better off. From Theorem 4, we know that as long as not all nodes deviate to zero, there exists a best response dynamic sequence for the system to go back to the navigable small world, and if all nodes move synchronously, the system reaches the navigable small world in at most $k \lfloor n/2 \rfloor$ synchronous steps. We now give a further result on the stability of navigable small-world in tolerating perturbations of random players: we show that even if each individual independently deviates to an arbitrary strategy with a fairly large probability, the system moves back to the navigable

small world in just one synchronous step, and even if players move asynchronously, it is guaranteed that the system moves back to the navigable small world after each node takes at least one asynchronous step.

THEOREM 7. *Consider the navigable small-world equilibrium $\mathbf{r} \equiv k$ for the DRB game in a k -dimensional grid ($k > 1$). Suppose that with probability p_u each node $u \in V$ independently perturbs r_u to an arbitrary strategy $r'_u \in \Sigma$, and with probability $1 - p_u$ $r'_u = r_u$. Then for any constant ε with $0 < \varepsilon < \gamma/4$, there exists $n_0 \in \mathbb{N}$ (depending only on k , γ , and ε), for all $n \geq n_0$, if $p_u \leq 1 - n^{-\varepsilon}$, with probability at least $1 - 1/n$, the perturbed strategy profile \mathbf{r}' moves back to the navigable small world ($\mathbf{r} \equiv k$) in one synchronous step, or as soon as every node takes at least one asynchronous step in the best response dynamic.*

PROOF (SKETCH). The independently selected deviation node set satisfies that with high probability, for any node u , at sufficiently many distance levels from u there are enough fraction of non-deviating nodes. We then show that u obtains higher order payoff just from these non-deviating nodes than any possible payoff she could get from any possible deviation. The detailed proof is in Appendix F. \square

Notice that the bound of $1 - n^{-\varepsilon}$ is close to 1 when n is sufficiently large, meaning that the navigable equilibrium tolerates arbitrary deviations from a large number of random nodes.

For the random small-world network, which is shown to be the other NE, Theorem 4 already implies that even one deviating player could possibly drive the system out of the random small-world equilibrium and lead it towards the navigable small-world equilibrium. However, converging to navigable small world is not guaranteed in this case. In the following, we show a stronger convergence result: if each individual u deviates from $r_u = 0$ independently with even a small probability, then the system could switch to the navigable small world in just one synchronous step, or after each node takes at least one asynchronous step, and the convergence to the navigable small world is guaranteed in this case.

THEOREM 8. *For the DRB game in a k -dimensional grid ($k > 1$) with the initial strategy profile $\mathbf{r} \equiv 0$ and a finite perturbed strategy set $S \subset \Sigma$ with at least one non-zero entry ($0 < \max S \leq \beta$), for any constant ε with $0 < \varepsilon < \gamma/2$, there exists $n_0 \in \mathbb{N}$ (depending only on k , γ , and ε), for all $n \geq n_0$, if for any $u \in V$, with independent probability of $p \geq n^{-\frac{(k-1)\varepsilon}{k+\beta}}$, $r_u \in S \setminus \{0\}$ after the perturbation, then with probability at least $1 - 1/n$, the network converges to the navigable small world in one synchronous step, or as soon as every node takes at least one asynchronous step in the best response dynamic.*

PROOF (SKETCH). We consider the gain of a node u when selecting $r_u = k$ separately from each group of nodes with the same strategy after the perturbation, and then apply the results in Theorem 1. The full proof is in Appendix G. \square

Note that $1/n^{\frac{(k-1)\varepsilon}{k+\beta}}$ is very small for large n and a finite perturbed strategy set S , which implies that the best response of any node u in the perturbed profile becomes $r_u = k$ as long as a small number of random nodes are perturbed to a finite set of nonzero strategies.

4.3 Implications from Theoretical Analysis

Combining the above theorems together, we obtain a better understanding of how the navigable small-world network is formed. From any arbitrary initial state, best response dynamic drives the system toward some equilibrium, with the navigable small world as one of them (Corollary 2 and Theorem 4). Even if the systems temporarily converges to a non-navigable equilibrium, the state will not be stable — either a small-size collusion (Theorem 6) or a small-size random perturbation (Theorem 8) would make the system leave the current equilibrium and quickly enter the navigable equilibrium. Once entering the navigable equilibrium, it is very hard for the system to move away from it — no collusion of any size would drive the system away from this equilibrium (Theorem 5), and even if a large random portion of nodes deviate arbitrarily the system still converge back to the navigable equilibrium as long as each node takes one best-response step (Theorem 7). These theoretical results strongly support that the navigable small world is the unique stable system state, which suggests that the fundamental balance between reaching out to remote people and seeking reciprocal relationship is crucial to the emergence of navigable small-world networks.

5. QUALITY OF EQUILIBRIA

In a Nash equilibrium, each user is maximizing its individual payoff. However, there is also a global function of social welfare, which is the total payoff of all nodes. A natural question then is how the social welfare of a system is affected when its users are selfish. Thus, in this section, we would like to examine how good the solution represented by an equilibrium is relative to the global optimum. To do this, we first examine the performance of the global optimum.

THEOREM 9. *In the k -dimensional DRB game, the optimal social welfare is $\Theta\left(\frac{n^{k+1}}{\ln^2 n}\right)$ for sufficiently large n .*

PROOF (SKETCH). We prove that a node u with $r_u = k$ could get high payoff if it has at least one neighbor v with $r_v > k$. In this case, the node u could get both large grid distance to long-range contacts and high reciprocity (at least from v). So if the system has a constant fraction of such nodes, the social welfare is optimized. The detailed proof is included in Appendix H. \square

The proof of the above theorem provides some interesting insights: First, the optimal strategy profile exhibits *inequality* in the distribution of payoff. The rich (e.g., those with strategy of k) could get high payoff whereas the poor (e.g., those with strategy larger than k) only get very low payoff. Furthermore, the optimum social welfare is achieved when the poor sacrifice their distance payoff and focus on their reciprocity (by selecting a strategy greater than k), so that their rich neighbors could obtain a high balanced payoff of both distance and reciprocity. This situation reminds us social relationships generated by different social status (e.g. employee-employer relationship) or by tight bound with mutual understanding and support (such as marriage relationship).

We next focus on the standard measures of the sub-optimality introduced by self-interested behavior. In particular, price of stability (PoS) is the ratio of the solution quality at the best Nash equilibrium relative to the global

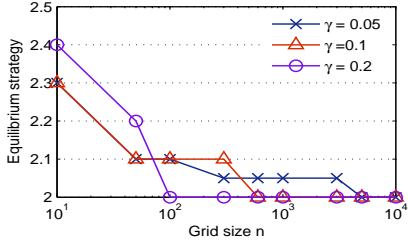


Figure 5: Equilibrium strategy in the 2D grid of different size and granularity.

optimum, whereas the price of anarchy (PoA) is the ratio of the worst Nash equilibrium to the optimum.

THEOREM 10. *In the k -dimensional DRB game, for sufficiently large n , the PoS is $\Theta(\ln n)$ and the PoA is $\Theta\left(\frac{n^k}{\ln^2 n}\right)$.*

PROOF (SKETCH). From the analysis in Section 4.1, we know that the system has only two Nash equilibria $\mathbf{r} \equiv k$ and $\mathbf{r} \equiv 0$, corresponding to navigable and random small-world networks, respectively. We show that navigable small-world network is a better equilibrium since the strategy of k provides the best balance between grid distance to long-range contacts and reciprocity. Combined with Theorem 9, we get the PoS and PoA of the system. The detailed proof is included in Appendix I. \square

The above theorem indicates that, in the good case when the system is in the navigable network equilibrium, the social welfare is reasonably close to the social optimum (with ratio $\Theta(\ln n)$ among n^k nodes), but in the bad case when the network is in the random network equilibrium, the social welfare is far from the social optimum.

6. EMPIRICAL EVALUATION

In this section, we empirically examine the stability of navigable small-world NE. We first simulate the DRB game on two dimensional grids, and consider nodes having full information, limited information, or no information of other players' strategies. We next simulate the game on Renren and Livejournal networks where population is not evenly distributed geographically.

Before the main empirical evaluation, we also test the effect of the grid size on navigable equilibrium, since our theoretical results require sufficiently large grids. Our theoretical analysis shows that one can find a large enough constant n_0 , such that the navigable equilibrium is exactly $\mathbf{r} \equiv 2$ for all $n \geq n_0$. Thus, we first verify empirically the relationship between the size of the grid and the actual connection preference value for the equilibrium. Figure 5 shows how the equilibrium value changes over n in a 2D grid, with various granularity. For example, with a granularity of $\gamma = 0.1$, the equilibrium decreases from $\mathbf{r} \equiv 2.3$ for a very small 10×10 grid, to $\mathbf{r} \equiv 2$ for a 1000×1000 grid. This shows that we do not need a very large grid in order to obtain results close to our theoretical predictions. In our following experiments, we use a 100×100 grid with the granularity $\gamma = 0.1$, which leads to an equilibrium $\mathbf{r} \equiv 2.1$ close to theoretical prediction while reducing the simulation cost.

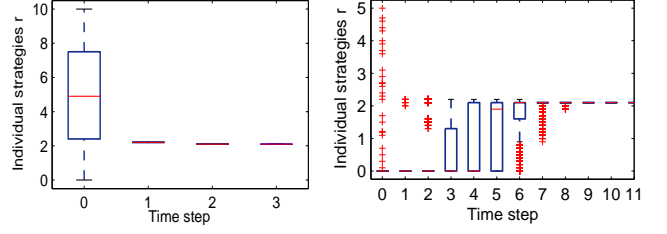


Figure 6: The return Figure 7: From ran- to navigable small-world dom NE to small-world NE (perturbed probabil- NE (perturbed probabil- ity $p=1$). ity $p=0.01$).

6.1 Stability of NE under Perturbation

To demonstrate the stability of navigable NE, we simulate the DRB game with random perturbation. At time step 0, each player is perturbed independently with probability p . If the perturbation occurs on a player u , we assume that the player u chooses a new strategy uniformly at random from the interval $[0, 10] \cap \Sigma$. Notice that for strategy $r_u > 10$, the behavior of nodes is similar to $r_u = 10$ as nodes only connect to the 4 grid neighbors. Let \mathbf{r}^0 be the strategy profile at time 0 after the perturbation. At each time step $t \geq 1$, every player picks the best strategy based on the strategies of others in the previous step.

$$r_u^t = \operatorname{argmax}_{r_u \in \Sigma \cap [0, 10]} \pi(r_u, \mathbf{r}_{-u}^{t-1}), \forall u, \forall t > 1.$$

Figure 6 shows an extreme case where every player is perturbed when the initial profile is $\mathbf{r} \equiv 2$. The box-plot shows the distribution of players' strategies at each step. The figure shows that in just two steps the system returns to the navigable small-world NE. We tested 100 random starting profiles, and all of them converge to the navigable NE within two steps. This simulation result indicates that the navigable NE is very stable for random perturbations.

To contrast, we study the stability of the random small-world network in terms of tolerating perturbations. Figure 7 shows the result of randomly perturbing only 1% of players at the random NE, which are shown as the outliers at step 0. Note that 1% perturbation does not meet the requirement in Theorem 8. However, this small fraction of players would affect the decision of additional players in their vicinity, who can significantly improve the reciprocity by also linking in the vicinity (indicated by Theorem 4). The figure clearly shows that in a few steps, more and more players would change their strategies, and the system finally goes to the navigable small-world NE.² We tested 100 random starting profiles, and all of them converge to the navigable NE within at most 12 steps.

These results show that the navigable small-world NE are robust to perturbations, while random small-world NE is not stable and easily transits to the small-world NE under a slight perturbation.

6.2 DRB Game with Limited Knowledge

²In step 1 and 2 in Figure 7, the number of outliers is larger than in step 0, even though the rendering make it seems they are less.

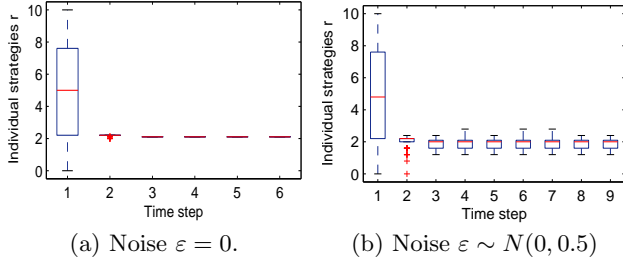


Figure 8: Network evolution where each player only knows the strategies of their friends.

Scenario 1: knowing friends’ strategies. In practice, a player does not know the strategies of all players. So we now consider a weaker scenario where a player only knows the strategies of their friends. With these limited knowledge, a player can guess the strategies of all other players and pick the best response to the estimated strategies of all players.

To examine the convergence of navigable small-world NE in this scenario, we simulate the DRB game as follows. At time step 0, each player chooses an initial strategy uniformly at random from the interval $[0, 10] \cap \Sigma$. At every step $t \geq 0$, each player u creates q out-going long-range links based on her current strategy r_u^t , and learns the connection preferences of these q long-range contacts. Let F_u^t be the set of these q long-range contacts. We further allows a random noise term ε for each connection preference learned from the friends. Let r_v^t ($v \in F_u^t$) be the learned (noisy) connection preference. Then based on these newly learned connection preferences, player u estimates the strategies of all other players. One reasonable estimation method is to assume that players close to one another in grid distance have similar strategy. More specifically, for a non-friend node $v \notin F_u^t$, u estimates the strategy of v by the average weight of known strategies:

$$r_v^t = \frac{\sum_{f \in F_u^t} r_{f,t-1} / d_M(v, f)}{\sum_{f \in F_u^t} 1 / d_M(v, f)}.$$

Here we do not use the connection preferences learned in the previous steps and effectively assume that those old links are removed. This is both for convenience, and also reasonable since people could only maintain a limited number of connections and it is natural that new connections replace the old ones. Moreover, the connection preferences of those old connections may become out-dated in practice anyway. After the estimation procedure, player u uses the strategy r_v^t from all other players (either learned or estimated) to compute its best response r_u^{t+1} for the next step.

In our experiment, we set $q = 30$. Figure 8(a) shows that when players have accurate knowledge of the strategies of their friends without noise, the system converges in just two steps. Even when the information on friends’ strategies is noisy, the system can still quickly stabilize in a few steps to a state close to the navigable small-world NE, as shown in Figure 8(b). We tested 100 random starting profiles and also other estimation methods such as randomly choosing a connection preference based on friends’ connection preference distributions, and results are all similar. This experiment further demonstrates the robustness of the small-world NE even under limited information on connection preferences.

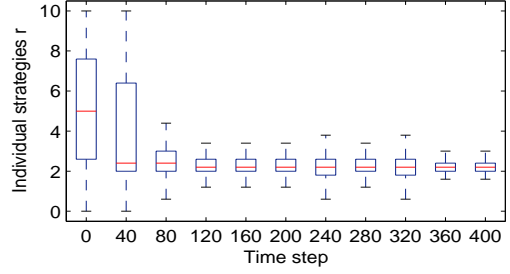


Figure 9: Network evolution where players have no knowledge of strategies of others.

Scenario 2: No information about others’ strategies.

We now consider the weakest scenario where each player has no knowledge about the strategies of other players. To get the payoff in this scenario, a player creates a certain number of links with the current strategy, and computes the payoff by multiplying the average link distance and the percentage of reciprocal links.

To make it even harder, we do not allow the player to try many different strategies at each step before fixing her strategy for the step. Instead, at each step each player only has one chance to slightly modify her current strategy. If the new strategy yields better payoff, the player would adopt the new strategy. So as the time goes on, the player could change the strategy towards the best one.

We simulate the DRB game as follows: At time step 0, each player chooses an initial strategy uniformly at random from the interval $[0, 10] \cap \Sigma$. Every player creates q out-going links with her current strategy. At each time step $t \geq 1$, each player changes the strategy, i.e., $r_u \leftarrow r_u + \delta$, and creates q new links with this new strategy, where δ is a random number determined as follows. First, for the sign of δ , in the first step it is randomly assigned positive or negative sign with equal probability; in the raining steps, to make the search efficient, we keep the sign of δ if the previous change leads to a higher payoff; otherwise we reverse the sign of δ . For the magnitude of δ , i.e. $|\delta|$, we sample a value uniformly at random from $(0, 1] \cap \Sigma$.

We simulate this system with $q = 30$. Figure 9 demonstrates that the system can still evolve to a state close to the navigable small-world NE in a few hundred steps, e.g., the strategies of 80.5% players fall in the interval $[1.8, 2.4]$, and the median of the strategies is the navigable NE strategy of 2.1. We test 50 random starting profiles, and take snapshots of the strategy profiles at the time step $t = 500$. On average, the strategies of 79.8% players in the snapshots fall in the interval $[1.8, 2.4]$.

In summary, our empirical evaluation strongly supports that our payoff function considering the balance between link distance and reciprocity naturally gives rise to the navigable small-world network. The convergence to navigable equilibrium will happen either when the players know all other players’ strategies, or only learn their friends’ strategies, or only use the empirical distance and reciprocity measure. Once in the navigable equilibrium, the system is very stable and hard to deviate by any random perturbation. Furthermore, other equilibria such as the random small world is not stable, in that a small perturbation will drive the system back to the navigable small-world network.

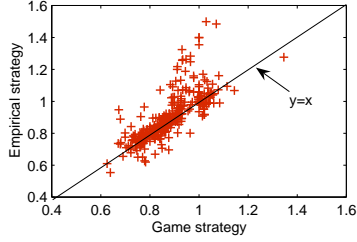


Figure 12: Game strategies vs. empirical strategies at city level in Renren network.

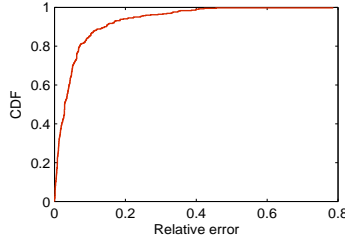


Figure 13: Relative error between game strategy and empirical strategy in Renren network.

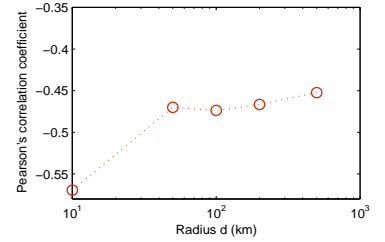


Figure 14: Correlation between the empirical strategy of a city and the number of people living in a d -km radius around this city in Renren.

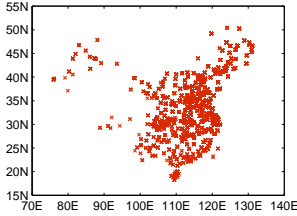


Figure 10: The home-town location of Renren users.

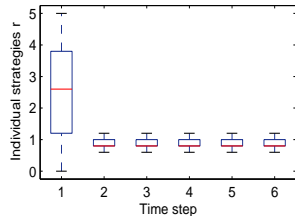


Figure 11: Simulated network work evolution over Renren grid.

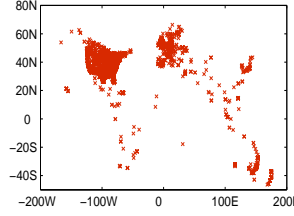


Figure 15: The home-town location of LiveJournal users.

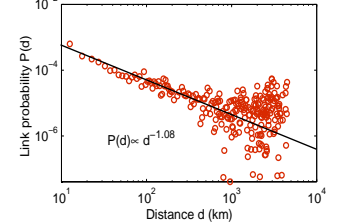


Figure 16: Friendship probability vs. distance in LiveJournal.

6.3 DRB Game under Real Networks

Recall that real network population is not evenly distributed geographically as in the Kleinberg's model. So we want to examine if our game could lead to an overall connection preference r similar to the empirical one in the real network. To do so, we examine our game with the following two real networks:

Renren Network. We sample 10K users at random from Renren network, and we construct a real grid through mapping the hometown listed in users' profiles to (longitude, latitude) coordinates, as shown in Figure 10. To examine the convergence of navigable small-world NE in this scenario, we simulate the DRB game as follows. at time step 0, each player chooses an initial strategy uniformly at random from the interval $[0, 5] \cap \Sigma$. At each time step $t \geq 1$, every player picks the best strategy based on the strategies of others in the previous step.

$$r_u^t = \operatorname{argmax}_{r_u \in \Sigma \cap [0, 5]} \pi(r_u, \mathbf{r}_{-u}^{t-1}), \forall u, \forall t > 1.$$

Figure 11 shows that in a few steps, the system reaches an NE, where individual users adopt their respective equilibrium strategies. In the NE, the mean of the strategies is 0.85 and the strategies of 88.1% users fall in the interval $[0.7, 1.1]$. So the overall connection preference of users in the simulated game is very close to the empirical value of 0.9 shown in Figure 2. As our game indicates that users might have various connection preferences under a non-uniform population distribution, it is interesting to examine whether our game captures the right preference heterogeneity depending on locations. Using the real data, we estimate the empirical connection preference of users living in the similar geographic

locations as follows. Recall that the resolution of our geographic data is limited to the level of towns and cities, so we compute the city-level friendship probability $p_C(d)$ for any given distance d and any given city C by the proportion of friendships among all pairs (u, v) with distance d and with u living in the city C . Then we fit the $p_C(d)$ with power-law distribution, and take the *exponent* of the fitted distribution as the empirical connection preference of users living in this city. Figure 12 shows the comparison between the connection preferences obtained in game and the empirical ones for a total of 442 cities in Renren, and Figure 13 plots the cumulative distribution function (CDF) of the relative error between game-driven strategies and the corresponding empirical ones. We see that our game also captures the right connection preference at city level, as indicated by a high Pearson's correlation coefficient of 0.70 and a low average relative error of 6.1%.

To further understand the impact of location on user linking behavior, Figure 14 shows the Pearson's correlation coefficient between a city's connection preference and the number of people living in a d -km radius around this city in Renren. We see that people's connection preference is negatively correlated with the number of people living in their vicinity — an individual with more people living in a 10km radius of tends to have a smaller connection preference (Pearson's correlation -0.57), and the negative correlation becomes weaker when the radius becomes larger. As our game generates the similar connection preferences of individual cities, it thus provides an explanation for the observed correlation. Specifically, in small towns, people should adopt a relatively large value of strategy of r for high reciprocity, because there are few nearby nodes providing reciprocal links and most

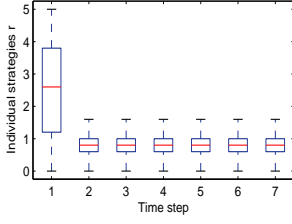


Figure 17: Simulated network evolution over LiveJournal network.

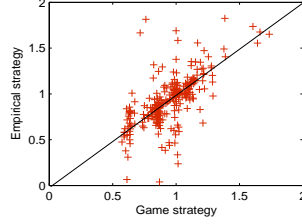


Figure 18: Game strategies vs. empirical strategies at city level in LiveJournal network.

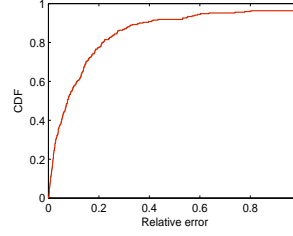


Figure 19: Relative error between game strategy and empirical strategy in LiveJournal network.

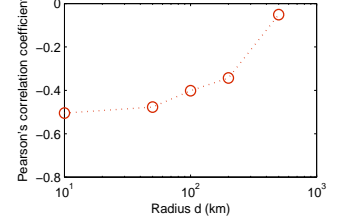


Figure 20: Correlation between the empirical strategy of a city and the number of people living in a d -km radius around this city in LiveJournal.

distant nodes are less likely to provide reciprocal links. In contrast, in large cities, people are less worried about reciprocity due to more nearby nodes, and thus they could adopt a relatively small value of strategy of r for better relationship distance. We also tested 100 random starting profiles, and results are all similar. The above result demonstrates that the DRB game could reach the navigable NE under real network, and furthermore, the game is able to explain the motivation behind the user's linking behavior in real social networks.

LiveJournal Network. To evaluate across-dataset generalization, we also examine our DRB game in the LiveJournal social network. LiveJournal is a community of bloggers with over 39 million registered users worldwide as the end of 2012. Each user provides a personal profile, including home location, personal interests and a list of other bloggers considered as friends. We obtain the profiles of 527,769 LiveJournal users through a crawl conducted from October 22 to October 28, 2015. Given the 224,155 users providing city information, we successfully obtained a meaningful geographic location for only 197,504 users, as shown in Figure 15. To get the empirical connection preference of these LiveJournal users, we compute the friendship probability $p(d)$ for any given distance d by the proportion of friendships among all pairs (u, v) with distance d . Figure 16 shows the relationship between friendship probability and geographic distance, which shows that the real connection preference of users is around 1.08.

Next, we repeat our game simulation in LiveJournal network as we do above in Renren network. Figure 17 shows that the system quickly reaches an NE, where the mean of the strategies is 0.94 and the strategies of 75.2% users fall in the interval $[0.6, 1.2]$. Figure 18 shows the comparison between the connection preferences obtained in game and the empirical ones for 270 cities with no less than 100 users, and Figure 19 plots the CDF of the relative error between game strategies and the corresponding empirical ones. We see that our game also captures the right connection preference at city level in LiveJournal, as indicated by a high Pearson's correlation coefficient of 0.64 and a low average relative error of 19.4%. Moreover, Figure 20 shows the Pearson's correlation coefficient between a city's connection preference and the number of people living in a d -km radius around this city in LiveJournal, which shows a similar trend as that of

Renren. These results demonstrate that our game does generalize across online social networks.

7. FUTURE WORK

Our study opens many possible directions of future work. For example, one may provide a theoretical analysis of the DRB game on the non-uniform population distributions, which has been empirically validated by our experiments on the Renren dataset. Another direction is to investigate deeper reasons or models on why individual's connection preference follows a power law form of $d_M(u, v)^{-r_u}$. Such studies may need to integrate prior studies on node and link dynamics and our game theoretic approach, and the integration may provide a more complete picture of the underlying mechanisms for navigable small-world networks. Moreover, our paper is the first study that unveils the important role of reciprocity in network navigability. We wish our study could encourage more empirical and theoretical studies on the relationship between reciprocity, distance, and navigability, and perhaps uncover the underlying human behavior model that integrates these factors and perhaps other factors together.

8. ACKNOWLEDGMENTS

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APPENDIX

A. COMMONLY USED RESULTS ON THE KLEINBERG'S SMALL WORLD AND THE DRB GAME

In all proofs in the appendix, for a given node $u \in V$, we denote $D(r_u) = \sum_{v \neq u} p_u(v, r_u) d_M(u, v)$ as its average grid distance of its long range contacts (simply referred to as the *link distance*), and $P_u(r_u, \mathbf{r}_{-u}) = \sum_{v \neq u} p_u(v, r_u) p_v(u, r_v)$ as its *reciprocity*. When $\mathbf{r}_{-u} \equiv s$, we simply use $P(r_u, s)$ to denote $P_u(r_u, \mathbf{r}_{-u} \equiv s)$. Moreover, for any $A \subseteq V$, let $P_{u,A}(\mathbf{r}) = \sum_{v \in A} p_u(v, r_u) p_v(u, r_v)$ be the reciprocity u obtained from subset A . We denote $c(r_u) = \sum_{v \neq u} d_M(u, v)^{-r_u}$ as u 's normalized coefficient. The subscript u in both $D(r_u)$, $P(r_u, s)$ and $c(r_u)$ is omitted because their values are the same for all $u \in V$.

Let n_D be the longest grid distance among nodes in $K(n, k, p, q, \mathbf{r})$. We have that $n_D = k \lfloor n/2 \rfloor$. We denote $b_u(j)$ as the number of players at grid distance j from u . We can find two constants ξ_k^- and ξ_k^+ only depending on the dimension k , so that $\xi_k^- j^{k-1} \leq b_u(j) \leq \xi_k^+ j^{k-1}$ for $1 \leq j \leq \lfloor n/2 \rfloor$ and $1 \leq b_u(j) \leq \xi_k^+ j^{k-1}$ for $\lfloor n/2 \rfloor < j \leq n_D$.³ Note that the payoff function (Eq. (3)) for the DRB game is indifferent of parameters p and q of the network, so we treat $p = q = 1$ for our convenience in the analysis.

We first show the following two lemmas, which will be used in the most of theorems.

LEMMA 2. *In the k -dimensional grid $K(n, k, p, q, \mathbf{r})$, for a given node $u \in V$ with a strategy of r_u , the normalized coefficient $c(r_u)$ has the following bounds:*

$$\begin{cases} c(r_u) \geq \frac{\xi_k^-}{2^{k+1}k} n^{k-r_u} & \text{if } r_u < k, \\ \frac{\xi_k^- \ln n}{2} \leq c(r_u) \leq \xi_k^+ \ln(2kn) & \text{if } r_u = k, \\ c(r_u) \geq \xi_k^- & \text{if } r_u > k. \end{cases}$$

PROOF. In the case of $r_u < k$, we write $\varepsilon = k - r_u$ ($\gamma \leq \varepsilon \leq k$). The coefficient $c(r_u)$ can be bounded as:

$$\begin{aligned} c(r_u) &= \sum_{v \neq u} d_M(u, v)^{-r_u} \geq \sum_{j=1}^{n/2} b_u(j) j^{-r_u} \geq \xi_k^- \sum_{j=1}^{n/2} j^{\varepsilon-1} \\ &\geq \xi_k^- \int_1^{n/2} x^{\varepsilon-1} dx \geq \frac{\xi_k^-}{\varepsilon} \left(\frac{n}{2} \right)^\varepsilon - \frac{\xi_k^-}{\varepsilon} \geq \frac{\xi_k^-}{2\varepsilon} \left(\frac{n}{2} \right)^\varepsilon \end{aligned}$$

The last inequality above relies on a loose relaxation of $\frac{1}{2} \left(\frac{n}{2} \right)^\varepsilon \geq 1$, which is guaranteed for all $n \geq 2^{1+1/\gamma}$ since $\varepsilon \geq \gamma$. Note that $\varepsilon < k$, so we have:

$$c(r_u) \geq \frac{\xi_k^-}{2^{1+\varepsilon} \varepsilon} n^\varepsilon \geq \frac{\xi_k^-}{2^{1+k} k} n^\varepsilon.$$

We now turn to the case of $r_u = k$. The upper bound of normalization coefficient $c(k)$ can be given as

$$c(k) = \sum_{v \neq u} d_M(u, v)^{-k} \leq \sum_{j=1}^{n_D} b_u(j) j^{-k} \leq \xi_k^+ \sum_{j=1}^{n_D} \frac{1}{j} \leq \xi_k^+ \ln(2kn),$$

and its lower bound is

$$c(k) \geq \xi_k \sum_{j=1}^{n/2} j^{-1} \geq \xi_k^- \int_1^{n/2} x^{-1} dx \geq \xi_k^- (\ln n - \ln 2) \geq \frac{\xi_k^- \ln n}{2}.$$

³The exact values of ξ_k^- and ξ_k^+ can be derived by the combinatorial problem of counting the number of ways to choose k non-negative integers such that they sum to a given positive integer j .

where the last inequality is true when $n \geq e^4$.

We finally consider the case of $r_u > k$, it is easily to get that

$$c(r_u) = \sum_{v \neq u} d_M(u, v)^{-r_u} \geq \sum_{j=1}^{n/2} b_u(j) j^{-r_u} \geq b_u(1) \geq \xi_k^-.$$

□

LEMMA 3. *In the k -dimensional grid $K(n, k, p, q, \mathbf{r})$, for a given node $u \in V$ with a strategy of r_u , the average distance of its long-range contacts $D(r_u)$ has the following bounds:*

$$\begin{cases} D(r_u) \leq \frac{\xi_k^+ k^{1+k}}{c(r_u)} n^{1+k-r_u} & \text{if } r_u < k, \\ \frac{\xi_k^- n}{2c(k)} \leq D(r_u) \leq \frac{\xi_k^+ n}{c(k)} & \text{if } r_u = k, \\ D(r_u) \leq \frac{\xi_k^+ k}{2\gamma c(r_u)} n^{1-\gamma} & \text{if } k < r_u < k+1, \\ D(r_u) \leq \frac{\xi_k^+}{c(r_u)} \ln(2kn). & \text{if } r_u \geq k+1. \end{cases} \quad (4)$$

PROOF. When $r_u < k$, we write $\varepsilon = k - r_u$ ($\gamma \leq \varepsilon \leq k$) and get the upper bound for the link distance

$$\begin{aligned} D(r_u) &= \frac{\sum_{j=1}^{n_D} b_u(j) \cdot j^{-r_u} \cdot j}{c(r_u)} \leq \frac{\xi_k^+ \int_1^{n_D+1} x^\varepsilon dx}{c(r_u)} \\ &\leq \frac{\xi_k^+ (n_D+1)^{1+\varepsilon}}{(1+\varepsilon)c(r_u)} \leq \frac{\xi_k^+ (kn)^{1+\varepsilon}}{c(r_u)} \leq \frac{\xi_k^+ k^{1+k}}{c(r_u)} n^{1+\varepsilon}. \end{aligned}$$

We now turn to the case of $r_u = k$. The upper bound of link distance $D(k)$ can be given as

$$D(r_u = k) = \frac{\sum_{j=1}^{n_D} b_u(j) \cdot j^{-k} \cdot j}{c(r_u)} \leq \frac{\xi_k^+ n}{c(k)},$$

and its lower bound is

$$D(r_u) \geq \frac{\sum_{j=1}^{n/2} b_u(j) \cdot j^{-k} \cdot j}{c(k)} \geq \frac{\xi_k^- n}{2c(k)}.$$

We finally consider the case of $r_u > k$. We write $\varepsilon = r_u - k$ ($\varepsilon \geq \gamma$), and the bound for the link distance is:

$$D(r_u) = \frac{\sum_{j=1}^{n_D} b_u(j) \cdot j^{-r_u} \cdot j}{c(r_u)} \leq \xi_k^+ \sum_{j=1}^{n_D} \frac{j^{-\varepsilon}}{c(r_u)} \leq \xi_k^+ \frac{1 + \int_1^{n_D} x^{-\varepsilon} dx}{c(r_u)}$$

In the case of $\varepsilon < 1$,

$$\begin{aligned} D(r_u) &\leq \xi_k^+ \frac{1 + \int_1^{n_D} x^{-\varepsilon} dx}{c(r_u)} \leq \frac{\xi_k^+}{(1-\varepsilon)c(r_u)} (kn/2)^{1-\varepsilon} \\ &\leq \frac{\xi_k^+ k}{2\gamma c(r_u)} n^{1-\varepsilon} \leq \frac{\xi_k^+ k}{2\gamma c(r_u)} n^{1-\gamma}. \end{aligned}$$

otherwise,

$$D(r_u) \leq \xi_k^+ \frac{1 + \int_1^{n_D} x^{-\varepsilon} dx}{c(r_u)} \leq \frac{\xi_k^+}{c(r_u)} \ln(2kn).$$

□

B. PROOF OF LEMMA 1

We first show the following key lemma, which will be used in later theorems too.

LEMMA 4. *In the k -dimensional DRB game, there exists a constant κ (only depending on model constants k and γ), for sufficiently large n (in particular $n \geq \max(e^4, 2k)$), the following statement holds: for any strategy profile \mathbf{r} , any node u with $r_u \neq k$, $\pi_u(r_u, \mathbf{r}_{-u}) \leq \kappa n^{1-\gamma}$.*

PROOF. We introduce some notations first. Given the strategy profile \mathbf{r} and a node u with $r_u \neq k$, we partition the rest nodes $V \setminus \{u\}$ into three sets: $V_{<k} = \{v \in V \setminus \{u\} \mid r_v < k\}$, $V_{>k} = \{v \in V \setminus \{u\} \mid r_v > k\}$, $V_{=k} = \{v \in V \setminus \{u\} \mid r_v = k\}$. Then we have

$$\pi_u(\mathbf{r}) = D(r_u) (P_{u,V_{<k}}(\mathbf{r}) + P_{u,V_{>k}}(\mathbf{r}) + P_{u,V_{=k}}(\mathbf{r})). \quad (5)$$

We now consider the case of $r_u < k$ and $r_u > k$ separately.

Payoff of $r_u < k$. Let $\varepsilon = k - r_u$ ($\gamma \leq \varepsilon \leq k$). We first consider the reciprocity player u obtains from the players in $V_{<k}$. We have $c(r_v) \geq c(k - \gamma)$ for $\forall v \in V_{<k}$, since $r_v \leq k - \gamma$. Then we have:

$$\begin{aligned} P_{u,V_{<k}}(\mathbf{r}) &= \sum_{v \in V_{<k}} \frac{d_M(u, v)^{-r_u - r_v}}{c(r_u)c(r_v)} \leq \sum_{v \in V_{<k}} \frac{d_M(u, v)^{-r_u}}{c(r_u)c(k - \gamma)} \\ &\leq \frac{\sum_{v \neq u} d_M(u, v)^{-r_u}}{c(r_u)c(k - \gamma)} = \frac{1}{c(k - \gamma)}. \end{aligned}$$

Combining the above inequality with the bounds given by Lemma 2 and Lemma 3, we get:

$$D(r_u)P_{u,V_{<k}}(\mathbf{r}) \leq \frac{\xi_k^+ 2^{k+3} k^{k+2}}{(\xi_k^-)^2} n^{1-\gamma}. \quad (6)$$

Next we examine the reciprocity that player u obtains from the players in $V_{>k}$. Note that for all $v \in V_{>k}$, $r_v \geq k + \gamma$. Using Lemma 2, we have:

$$\begin{aligned} P_{u,V_{>k}}(\mathbf{r}) &= \sum_{v \in V_{>k}} \frac{d_M(u, v)^{-r_u - r_v}}{c(r_u)c(r_v)} \\ &\leq \frac{\sum_{j=1}^{n_D} b_u(j) \cdot j^{-r_u} \cdot j^{-k-\gamma}}{\xi_k^- c(r_u)} = \frac{\xi_k^+ \sum_{j=1}^{n_D} j^{-1-r_u-\gamma}}{\xi_k^- c(r_u)} \\ &\leq \frac{\xi_k^+ (1 + \int_1^{n_D} x^{-1-r_u-\gamma} dx)}{\xi_k^- c(r_u)} \leq \frac{\xi_k^+ (1 + r_u + \gamma)}{\xi_k^- (r_u + \gamma) c(r_u)} \leq \frac{\xi_k^+ (k+1)}{\xi_k^- \gamma c(r_u)}. \end{aligned}$$

Based on the bounds in Lemma 2 and Lemma 3, we get:

$$\begin{aligned} D(r_u)P_{u,V_{>k}}(\mathbf{r}) &\leq \frac{\xi_k^+ 2^{2k+2} k^2 (k+1) k^{1+k}}{(\xi_k^-)^{2\gamma}} n^{1-\varepsilon} \\ &\leq \frac{\xi_k^+ 2^{2k+3} k^{k+4}}{(\xi_k^-)^{2\gamma}} n^{1-\gamma}. \end{aligned} \quad (7)$$

We now examine the payoff of player u from players in $V_{=k}$. In this case, the upper bound for the reciprocity is:

$$P(r_u, k) = \frac{\sum_{j=1}^{n_D} b_u(j) \cdot j^{-r_u} \cdot j^{-k}}{c(r_u)c(k)} \leq \xi_k^+ \sum_{j=1}^{n_D} \frac{j^{\varepsilon-1-k}}{c(r_u)c(k)}.$$

Notice that $\varepsilon \leq k$, we have:

$$\begin{aligned} P(r_u, k) &\leq \frac{\xi_k^+}{c(r_u)c(k)} \left(1 + \int_1^{n_D} j^{\varepsilon-1-k} \right) \\ &\leq \begin{cases} \frac{\xi_k^+ (1+k-\varepsilon)}{(k-\varepsilon)c(r_u)c(k)} \leq \frac{2k\xi_k^+}{\gamma c(r_u)c(k)} & \text{if } \varepsilon < k, \\ \frac{2\xi_k^+ \ln(2kn)}{c(r_u)c(k)} & \text{if } \varepsilon = k, \end{cases} \end{aligned}$$

The inequalities in the cases above use the facts $\gamma \leq \varepsilon \leq k$ and $k - \varepsilon \geq \gamma$ when $\varepsilon < k$.

Combing the the above bounds on reciprocity with bounds given by Lemma 2 and Lemma 3, we have the payoff of node

u getting from $V_{=k}$:

$$\begin{aligned} D(r_u)P_{u,V_{=k}}(\mathbf{r}) &\leq D(r_u)P(r_u, k) \\ &\leq \begin{cases} \frac{(\xi_k^+)^2 2^{2k+4} k^{k+3} \ln(2kn)}{(\xi_k^-)^3} n^{1-\gamma} \leq \frac{(\xi_k^+)^2 2^{2k+5} k^{k+3}}{(\xi_k^-)^3} n^{1-\gamma} & \text{if } \varepsilon = k. \\ \frac{(\xi_k^+)^2 2^{2k+4} k^{k+4}}{\gamma (\xi_k^-)^3} \frac{n^{1-\gamma}}{\ln n} & \text{if } \varepsilon < k. \end{cases} \end{aligned} \quad (8)$$

The last inequality in the case of $\varepsilon = k$ requires $n \geq 2k$.

Adding up results in Eq.(6), (7), (8), we obtain that

$$\pi(r_u, \mathbf{r}_{-u}) \leq \frac{3(\xi_k^+)^2 \cdot 2^{2k+5} k^{k+4}}{\gamma (\xi_k^-)^3} n^{1-\gamma} \leq \frac{(\xi_k^+)^2 2^{2k+7} k^{k+4}}{\gamma (\xi_k^-)^3} n^{1-\gamma}, \quad (9)$$

when $n \geq \max\{e^4, 2k\}$.

Payoff of $r_u > k$. Let $\varepsilon = r_u - k$ ($\varepsilon \geq \gamma$). For this case, we can relax the reciprocity $P_u(r_u, \mathbf{r}_{-u})$ to one and only upper bound link distance $D(r_u)$. Applying bounds in Lemma 2 and Lemma 3, we obtain:

$$\pi(r_u = k + \varepsilon, \mathbf{r}_{-u}) \leq \begin{cases} \frac{\xi_k^+ k}{\xi_k^- \gamma} n^{1-\gamma} & \text{if } \varepsilon < 1, \\ \frac{\xi_k^+}{\xi_k^-} \ln(2kn) \leq 2n^{1-\gamma} & \text{if } \varepsilon \geq 1. \end{cases} \quad (10)$$

The last inequality in the above case of $\varepsilon \geq 1$ holds when $n \geq 2k$ and $\gamma \leq 1/2$.

Finally, the lemma holds when we combine Eq.(9) and (10) \square

PROOF OF LEMMA 1. For a given node w , define the set of nodes with distance of δ to w as: $N_{w,\delta} = \{u \mid u \in V \wedge d_M(u, w) \leq \delta\}$.

In the case of $r_w \geq k$, for any $u \in N_{w,\delta}$, if u chooses the strategy $r_u = k$, we have:

$$P(r_u, \mathbf{r}_{-u}) > p_u(w, r_u) p_w(u, r_w) \geq \frac{d_M(u, w)^{-2k}}{c(k)^2} \geq \frac{\delta^{-2k}}{c(k)^2}.$$

Combining the above inequality with the bounds in Lemma 2 and Lemma 3, we get:

$$\pi(r_u = k, \mathbf{r}_{-u}) > \frac{\xi_k^- \delta^{-2k} n}{2c^3(k)} \geq \frac{\xi_k^- \delta^{-2k}}{2(\xi_k^+)^3} \frac{n}{\ln^3(2kn)}. \quad (11)$$

However, if node u chooses $r_u \neq k$, by Lemma 4 we know that there is a constant κ such that for all sufficiently large n , $\pi(r_u, \mathbf{r}_{-u}) \leq \kappa n^{1-\gamma}$. We see that the payoff of $r_u = k$ is in strictly higher order in n than its original payoff, thus there exists $n_0 \in \mathbb{N}$ (n_0 may depend on δ), such that for all $n \geq n_0$, $r_u = k$ is the unique best response to \mathbf{r}_{-u} for any $u \in N_{w,\delta}$.

In the case of $r_w = \max_{v \in V} r_v$, if $r_w \geq k$, from the above analysis we know that $r_u = k$ is the unique best response to \mathbf{r}_{-u} for any $u \in N_{w,\delta}$.

Otherwise, given $r_w < k$, we know $V = V_{<k}$. In this case, we further partition the nodes $V_{<k}$ into two sets: $V_{>0} = \{v \in V \mid k > r_v > 0\}$ and $V_{=0} = \{v \in V \mid r_v = 0\}$. So we know that:

$$\pi_u(\mathbf{r}) = D(r_u) (P_{u,V_{>0}}(\mathbf{r}) + P_{u,V_{=0}}(\mathbf{r})). \quad (12)$$

Let r_{min} be the minimum value among the strategies of users in the set of $V = V_{>0}$. Clearly, $\gamma \leq r_{min} \leq r_w < k$.

Payoff of $r_u < k$. For any node $u \in N_{w,\beta}$, if it chooses $r_u < k$, let $\varepsilon = k - r_u$ ($\gamma \leq \varepsilon \leq k$). Notice that $c(v) \geq c(r_w)$

for any node $v \in V \setminus \{u\}$, so we have:

$$\begin{aligned} P_{u,V_{>0}}(\mathbf{r}) &= \sum_{v \in V_{>0}} \frac{d_M(u,v)^{-r_u-r_v}}{c(r_u)c(r_v)} \\ &\leq \sum_{v \in V \setminus \{u\}} \frac{d_M(u,v)^{-r_u-r_{\min}}}{c(r_u)c(r_w)} \\ &\leq \frac{\sum_{j=1}^{n_D} \xi_k^+ j^{\varepsilon-1-r_{\min}}}{c(r_u)c(r_w)}. \end{aligned}$$

If $r_{\min} + 1 \leq \varepsilon$, we have:

$$\begin{aligned} P_{u,V_{>0}}(\mathbf{r}) &\leq \xi_k^+ \int_{j=1}^{n_D+1} \frac{j^{\varepsilon-r_{\min}-1}}{c(r_u)c(r_w)} \leq \frac{\xi_k^+(n_D+1)^{\varepsilon-r_{\min}}}{(\varepsilon-r_{\min})c(r_u)c(r_w)} \\ &\leq \frac{\xi_k^+(kn)^{\varepsilon-r_{\min}}}{c(r_u)c(r_w)}. \end{aligned}$$

If $r_{\min} + 1 > \varepsilon$, we have:

$$\begin{aligned} P_{u,V_{>0}}(\mathbf{r}) &\leq \frac{\xi_k^+}{c(r_u)c(r_w)} \left(1 + \int_{j=1}^{n_D} j^{\varepsilon-1-r_{\min}}\right) \\ &\leq \begin{cases} \frac{\xi_k^+((kn)^{\varepsilon-r_{\min}} + \varepsilon - r_{\min} - 1)}{(\varepsilon - r_{\min})c(r_u)c(r_w)} & \text{if } r_{\min} \neq \varepsilon, \\ \frac{\xi_k^+ \ln(kn)}{c(r_u)c(r_w)} & \text{if } r_{\min} = \varepsilon. \end{cases} \end{aligned}$$

Combining the above bounds on reciprocity with bounds in Lemma 2 and Lemma 3, we have the payoff of node u gets from $V_{>0}$:

$$\begin{aligned} D(r_u)P_{u,V_{>0}}(\mathbf{r}) &\leq \begin{cases} \frac{(\xi_k^+)^2 2^{3k+3} k^{2k+4}}{(\xi_k^-)^{3\gamma}} n^{1+r_w-\gamma-k}, & \text{if } \varepsilon > r_{\min}, \\ \frac{(\xi_k^+)^2 2^{3k+3} k^{k+1}}{(\xi_k^-)^3} n^{1+r_w-\gamma-k} \ln(kn), & \text{if } \varepsilon = r_{\min}, \\ \frac{(\xi_k^+)^2 2^{3k+3} k^{k+4}}{(\xi_k^-)^{3\gamma}} n^{1+r_w-\gamma-k}, & \text{if } \varepsilon < r_{\min}. \end{cases} \end{aligned} \quad (13)$$

We now consider the payoff of a node u from the set $V_{=0}$. We have:

$$\begin{aligned} P_{u,V_{=0}}(\mathbf{r}) &= \sum_{v \in V_{=0}} \frac{d_M(u,v)^{-r_u}}{c(r_u)c(0)} \\ &\leq \sum_{v \in V \setminus \{u\}} \frac{d_M(u,v)^{-r_u}}{c(r_u)c(0)} = \frac{1}{c(0)}. \end{aligned} \quad (14)$$

It is easy to get:

$$c(0) = n^k - 1 \geq n^k/2. \quad (15)$$

Thus, combining with bounds in Lemma 2 and Lemma 3, we have the payoff of node u gets from $V_{=0}$:

$$D(r_u)P_{u,V_{=0}}(\mathbf{r}) \leq \frac{\xi_k^+(kn)^{1+\varepsilon}}{c(r_u)c(0)} \leq \frac{2^{k+2} k^{2+k} \xi_k^+}{\xi_k^-} n^{1-k}. \quad (16)$$

Combining the above bounds in Eq.(13) and Eq.(16) with Eq.(12), we see the payoff of $r_u < k$ in \mathbf{r} is at most $O(n^{1+r_w-\gamma-k} \ln(kn))$.

Payoff of $r_u > k$. If node u chooses $r_u > k$, let $\varepsilon = r_u - k \geq \gamma$. We have:

$$\begin{aligned} P_{u,V_{>0}} &= \sum_{v \in V_{>0}} \frac{d_M(u,v)^{-r_u-r_v}}{c(r_u)c(r_v)} \leq \frac{\sum_{v \in V_{>0}} d_M(u,v)^{-r_u-r_v}}{c(r_u)c(r_w)} \\ &< \frac{\sum_{v \in V \setminus \{u\}} d_M(u,v)^{-r_u}}{c(r_u)c(r_w)} = \frac{1}{c(r_w)}. \end{aligned}$$

Combining the above inequality with Lemma 2 and Lemma 3, we have the payoff of node u gets from $V_{>0}$:

$$D(r_u)P_{u,V_{>0}} = \begin{cases} \frac{2^{k+1} k^2 \xi_k^+}{(\xi_k^-)^{2\gamma}} n^{1+r_w-k-\gamma} & \text{if } \varepsilon < 1, \\ \frac{2^{k+1} k \xi_k^+}{(\xi_k^-)^2} \frac{\ln(2kn)}{n^{k-r_w}} & \text{if } \varepsilon \geq 1. \end{cases} \quad (17)$$

Combining with bounds in Lemma 2 and Lemma 3 with Eq.(14) and Eq.(15), we have the payoff of node u gets from $V_{=0}$:

$$D(r_u)P_{u,V_{=0}} = \begin{cases} \frac{4k \xi_k^+}{\xi_k^-} n^{1-k-\gamma} & \text{if } \varepsilon < 1, \\ \frac{2 \xi_k^+}{\xi_k^-} \frac{\ln(2kn)}{n^k} & \text{if } \varepsilon \geq 1. \end{cases} \quad (18)$$

Combining the above bounds in Eq.(17) and Eq.(18) with Eq.(12), we see the payoff of $r_u > k$ is at most $O(n^{1+r_w-\gamma-k})$.

Payoff of $r_u = k$. However, if the node u chooses the strategy $r_u = k$, we have:

$$P(r_u) > p_u(v, r_u) p_w(u, r_w) > \frac{d(u, w)^{-k-r_w}}{c(k)c(r_w)} > \frac{\delta^{-2k}}{c(k)c(r_w)}.$$

Combining the above inequality with the bounds in Lemma 2 and Lemma 3, we get:

$$\pi(r_u = k, \mathbf{r}_{-u}) > \frac{\xi_k^- \delta^{-2k} n}{2c^2(k)c(r_w)} \geq \frac{k 2^k \delta^{-2k}}{(\xi_k^+)^2} \frac{n^{1+r_w-k}}{\ln^2(2kn)}. \quad (19)$$

We see that the payoff of $r_u = k$ is in strictly higher order in n than the payoff of $r_u < k$ or $r_u > k$, thus there exists $n_0 \in \mathbb{N}$ (which may depend on δ but do not depend on r_w since n^{1+r_w-k} is a common term), for all $n \geq n_0$, $r_u = k$ is the best response to \mathbf{r}_{-u} for any $u \in N_{w,\delta}$. \square

C. PROOF OF RANDOM SMALL WORLD EQUILIBRIUM

PROOF OF THEOREM 1 FOR $B_u(\mathbf{r}_{-u} \equiv s) = 0$ IF $s = 0$. When other players choose strategy $\mathbf{r}_{-u} \equiv 0$, the reciprocity of the player u is constant:

$$P(r_u, \mathbf{r}_{-u} \equiv 0) = \frac{\sum_{v \neq u} p_u(v, r_u)}{|V| - 1} = \frac{1}{|V| - 1}. \quad (20)$$

Thus, the payoff of the player u is only determined by the link distance $D(r_u)$. Let $X(r_u)$ be the random variable denoting the grid distance from u 's long-range contact to u . Then we have $D(r_u) = E[X(r_u)]$. We want to show the following intuitive claim:

Claim 1. For any $r_u < r'_u$, $X(r_u)$ strictly stochastically dominates $X(r'_u)$, i.e., for all $1 \leq \ell < n_D$, $\Pr(X(r_u) \leq \ell) < \Pr(X(r'_u) \leq \ell)$.

Proof of the claim. Let $q(r_u, j)$ be the probability that u 's long-range contact is a particular node v at grid distance j from u . By definition, $q(r_u, j) = j^{-r_u}/c(r_u)$. Then we have $\frac{q(r_u, j+1)}{q(r_u, j)} = \left(\frac{j+1}{j}\right)^{-r_u}$. Thus $q(r_u, j)$ is non-increasing in j , and the decreasing ratio is faster when r_u is larger. Since we know that $\sum_{j=1}^n q(r_u, j) b_u(j) = 1$, it must be that $q(r_u, 1) < q(r'_u, 1)$, $q(r_u, n_D) > q(r'_u, n_D)$, and there exists a \bar{j} such that for all $j \leq \bar{j}$, $q(r_u, j) \leq q(r'_u, j)$, and for all $j > \bar{j}$, $q(r_u, j) > q(r'_u, j)$.

By the definition of $X(r_u)$, we have $\Pr(X(r_u) \leq \ell) = \sum_{j=1}^{\ell} q(r_u, j) b_u(j)$. Thus, for any $1 \leq \ell \leq \bar{j}$, $\Pr(X(r_u) \leq$

$\ell) = \sum_{j=1}^{\ell} q(r_u, j) b_u(j) < \sum_{j=1}^{\ell} q(r'_u, j) b_u(j) = \Pr(X(r'_u) \leq \ell)$. For any $\bar{j} < \ell < n_D$, $\Pr(X(r_u) \leq \ell) = \sum_{j=1}^{\ell} q(r_u, j) b_u(j) = 1 - \sum_{j=\ell+1}^{n_D} q(r_u, j) b_u(j) < 1 - \sum_{j=\ell+1}^{n_D} q(r'_u, j) b_u(j) = \Pr(X(r'_u) \leq \ell)$. Therefore, we have the claim that $X(r_u)$ strictly stochastically dominates $X(r'_u)$.

With this claim, we immediately have $E[X(r_u)] > E[X(r'_u)]$. As a consequence, $D(0) = E[X(0)] > E[X(r'_u)] = D(r'_u)$ for any $r'_u > 0$. Therefore, $r_u = 0$ is the player u 's unique best response to $\mathbf{r}_{-u} \equiv 0$. \square

D. PROOF OF THEOREM 5

To prove the theorem, we first introduce the following lemma.

LEMMA 5. *In the k -dimensional DRB game, for sufficiently large n , the payoff of every node $u \in V$ in the navigable small world $\mathbf{r} \equiv k$ has the following bounds:*

$$\frac{(\xi_k^-)^2}{2(\xi_k^+)^3} \frac{n}{\ln^3(2kn)} \leq \pi(r_u = k, \mathbf{r}_{-u} \equiv k) \leq \frac{16(\xi_k^+)^2}{(\xi_k^-)^3} \cdot \frac{n}{\ln^3 n}. \quad (21)$$

PROOF. We have the lower bound for the reciprocity:

$$P(r_u, k) \geq \frac{\sum_{j=1}^{n/2} b_u(j) \cdot j^{-2k}}{c^2(k)} \geq \frac{\xi_k^-}{c^2(k)}.$$

Combining the above inequality with bounds given by Lemma 2 and Lemma 3, we get:

$$\pi(r_u = k, \mathbf{r}_{-u} \equiv k) \geq \frac{(\xi_k^-)^2}{2(\xi_k^+)^3} \frac{n}{\ln^3(2kn)}.$$

The upper bound on the reciprocity is:

$$\begin{aligned} P(r_u, \mathbf{r}_{-u} \equiv k) &= \frac{\sum_{j=1}^{n_D} b_u(j) \cdot j^{-2k}}{c^2(k)} \\ &\leq \frac{\xi_k^+}{c^2(k)} \left(1 + \int_{j=1}^{n_D} j^{-k-1}\right) \leq \frac{2\xi_k^+}{c^2(k)} \end{aligned}$$

Combining the above inequality with Lemma 2 and Lemma 3, we get the upper bound on the payoff:

$$\pi(r_u = k, \mathbf{r}_{-u} \equiv k) \leq \frac{2\xi_k^+ D(r_u = k)}{c^2(k)} \leq \frac{16(\xi_k^+)^2}{(\xi_k^-)^3} \cdot \frac{n}{\ln^3 n}.$$

\square

PROOF OF THEOREM 5. We actually prove a slightly stronger result: any node u in any strategy profile \mathbf{r} with $r_u \neq k$ is strictly worse off than its payoff in the navigable equilibrium, when n is large enough.

With the Lemma 5, we see that a player u has the payoff at least $\Omega\left(\frac{n}{\ln^3(2kn)}\right)$ before derivation. Suppose that a coalition C deviates, and the new strategy profile is \mathbf{r} . Then some node $u \in C$ must select a new $r_u \neq k$. By Lemma 4, there is a constant κ such that for all sufficiently large n , $\pi(r_u, \mathbf{r}_{-u}) \leq \kappa n^{1-\gamma}$. Thus we see that the payoff of u before the deviation is in strictly higher order in n than its payoff after the deviation. Therefore, for all sufficiently large n , u is strictly worse off, which means no coalition could make some member strictly better off while others not worse off. Hence, navigable small-world network ($\mathbf{r} \equiv k$) is a strong Nash equilibrium. \square

E. PROOF OF THEOREM 6

PROOF OF THEOREM 6. Given a pair of grid neighbors (u, v) , if they both choose the strategy k , we have:

$$P(r_u, \mathbf{r}_{-u}) > p_u(v, r_u) p_v(u, r_u) \geq \frac{d_M(u, v)^{-2k}}{c(k)^2} \geq \frac{1}{c(k)^2}.$$

Combining the above inequality with the bounds in Lemma 2 and Lemma 3, we get:

$$\pi(r_u = k, \mathbf{r}_{-u}) > \frac{\xi_k^- n}{2c^3(k)} \geq \frac{\xi_k^-}{2(\xi_k^+)^3} \frac{n}{\ln^3(2kn)}. \quad (22)$$

However, if node u chooses $r_u \neq k$, by Lemma 4 we know that there is a constant κ such that for all sufficiently large n , $\pi(r_u, \mathbf{r}_{-u}) \leq \kappa n^{1-\gamma}$. We see that the payoff of $r_u = k$ is in strictly higher order in n than its original payoff. Notice the payoff of node v is similar to node u , so both colluding node get strictly higher payoff. The theorem is proved. \square

F. PROOF OF THEOREM 7

FACT 1. (Chernoff bound). Let X be a sum of n independent random variables $\{X_i\}$, with $E[X_i] = \mu$; $X_i \in \{0, 1\}$ for all $i \leq n$. For any $0 < \epsilon < 1$,

$$\Pr[X \leq (1 - \epsilon)\mu] \leq e^{-\frac{\epsilon^2 \mu}{2}}, \Pr[X \geq (1 + \epsilon)\mu] \leq e^{-\frac{\epsilon^2 \mu}{3}}.$$

Based on the Chernoff bound, we have the following lemma. Let $Y_u(j, s)$ be the number of players with grid distance j to u and a strategy of s .

LEMMA 6. *In the k -dimensional DRB game ($k > 1$), for any $\eta > 0$, if each player chooses a strategy s independently with probability $p_s \geq \eta$ from a finite strategy set $S \subseteq \Sigma$, then for all $n \geq |S|$, with probability $1 - 1/n$, the following property holds:*

$$Y_u(j, s) > \frac{\eta b_u(j)}{2}, \forall u \in V, \forall s \in S, \forall j \in \mathbb{N} \left[\rho \left(\frac{\ln n}{\eta} \right)^{\frac{1}{k-1}}, \frac{n}{2} \right],$$

where $\rho = \left(\frac{24+8k}{\xi_k^-} \right)^{\frac{1}{k-1}}$ is a constant.

PROOF. Since individual players choose strategy of s independently with probability p_s , $E[Y_u(j, s)] = p_s b_u(j) \geq \eta b_u(j)$. Based on the Chernoff bound, we have:

$$\begin{aligned} P(Y_u(j) \leq (1 - \epsilon)E[Y_u(j, s)]) &\leq \exp\left(-\frac{\epsilon^2 E[Y_u(j, s)]}{2}\right) \\ &\leq \exp\left(-\frac{\epsilon^2 \eta b_u(j)}{2}\right). \end{aligned}$$

Note that $b_u(j) \geq \xi_k^- j^{k-1}$ for $0 < j \leq \lfloor n \rfloor / 2$. Let $m = |S|$. Let $\varrho = \left(\frac{(16+8k) \ln n + 8 \ln m}{\eta \xi_k^-} \right)^{\frac{1}{k-1}}$. For $\varrho \leq j \leq \lfloor n \rfloor / 2$, we have:

$$P\left(Y_u(j) \leq \frac{\eta b_u(j)}{2}\right) \leq \frac{1}{m n^{k+2}}.$$

Since there are n^k players in the k dimensional grid, by union bound, we have $\forall u, \forall s$, for any $\varrho \leq j \leq \lfloor n \rfloor / 2$,

$$P\left(Y_u(j) > \frac{\eta b_u(j)}{2}\right) \geq 1 - \frac{1}{n},$$

As m is a constant, we can rewrite ϱ as:

$$\varrho = \left(\frac{(16+8k)\ln n + 8\ln m}{\eta \xi_k^-} \right)^{\frac{1}{k-1}} \leq \left(\frac{24+8k}{\xi_k^-} \right)^{\frac{1}{k-1}} \left(\frac{\ln n}{\eta} \right)^{\frac{1}{k-1}},$$

holds for all $n \geq m$. \square

PROOF OF THEOREM 7. Given a deviation probability $p_u \leq 1 - n^{-\varepsilon}$ for node u , we know that node u still uses the original strategy k with a probability of $1 - p_u \geq n^{-\varepsilon}$. By Lemma 6 we know that, with probability $1 - \frac{1}{n}$, the following property holds:

$$Y_u(j, k) > \frac{n^{-\varepsilon} b_u(j)}{2}, \forall u \in V, \forall j \in \mathbb{N} \cap \left[\rho \left(\frac{\ln n}{n^{-\varepsilon}} \right)^{\frac{1}{k-1}}, \frac{n}{2} \right]. \quad (23)$$

When the above property holds, we fix any node u and examine its payoff. In the case of $r_u = k$, the reciprocity that u gets from those still choosing strategy of k is:

$$\begin{aligned} P_{u, V_k}(\mathbf{r}) &\geq \frac{n^{-\varepsilon}}{2} \frac{\sum_{j=\rho \left(\frac{\ln n}{n^{-\varepsilon}} \right)^{\frac{1}{k-1}}}^{n/2} b_u(j) \cdot j^{-2k}}{c^2(k)} \\ &\geq \frac{\xi_k^- n^{-\varepsilon}}{2} \frac{\sum_{j=\rho \left(\frac{\ln n}{n^{-\varepsilon}} \right)^{\frac{1}{k-1}}}^{n/2} j^{-k-1}}{c^2(k)} \\ &\geq \frac{\xi_k^- n^{-\varepsilon}}{2} \rho^{\frac{-(k+1)}{k-1}} \left(\frac{\ln n}{n^{-\varepsilon}} \right)^{\frac{-(k+1)}{k-1}} \frac{1}{c^2(k)} \\ &\geq \frac{\xi_k^- n^{-4\varepsilon}}{2c^2(k)\rho^3 \ln^3 n}. \end{aligned}$$

The last inequality holds as $k \geq 2$.

Combing with the above bound with bounds in Lemma 2 and Lemma 3, we get:

$$\pi_u(r_u = k, \mathbf{r}) \geq D(r_u) P_{u, V_k}(\mathbf{r}) \geq \frac{(\xi_k^-)^2}{2(\xi_k^+)^3 \rho^3} \frac{n^{1-4\varepsilon}}{\ln^6(2kn)}. \quad (24)$$

By Lemma 4, there is a constant κ such that for all sufficiently large n , $\pi(r_u \neq k, \mathbf{r}_{-u}) \leq \kappa n^{1-\gamma}$. Comparing with Eq. (24), since $\varepsilon < \gamma/4$, the payoff of u with strategy $r_u = k$ is in strictly higher order in n than its payoff after the deviation. Therefore, when the property Eq. (23) holds, for all sufficiently large n , u get strictly better payoff than any other strategy choice by choosing $r_u = k$ after the deviation.

Therefore, when the property Eq. (23) holds, the perturbed strategy profile \mathbf{r}' moves back to the navigable small world ($\mathbf{r} \equiv k$) in one synchronous step, as every player u moves from its current strategy to its best response $r_u = k$. Also, it is clear that the property Eq. (23) consistently holds as any player takes one asynchronous step. This is because the asynchronous move only increases the number of nodes choosing the strategy of k , so the best response of every player is always k after every asynchronous step. Thus, the perturbed strategy profile moves back to the navigable small world as soon as every node takes at least one asynchronous step. Notice that the property Eq. (23) holds with a probability of $1 - 1/n$, so the theorem is proved. \square

G. PROOF OF THEOREM 8

PROOF OF THEOREM 8. Fix any node $u \in V$. Let \mathbf{r} be the strategy profile after perturbation. We partition nodes

in $V \setminus \{u\}$ into sets $V_s, s \in S \cup \{0\}$, where $V_s = \{v \in V \setminus \{u\} \mid r_v = s\}$. Let $P_{u, V_s}(\mathbf{r})$ be the reciprocity u obtained from subset V_s . Then we have

$$\pi(r_u, \mathbf{r}_{-u}) = D(r_u) \cdot \sum_{s \in S \cup \{0\}} P_{u, V_s}(\mathbf{r}). \quad (25)$$

For any node u and any given $s \in S \cup \{0\}$, we now compare the payoff it gets from V_s when using $r_u = k$ and $r_u = s' \neq k$, respectively.

$$\frac{D(r_u = s') P_{u, V_s}(\mathbf{r})}{D(r_u = k) P_{u, V_s}(\mathbf{r})} = \frac{D(r_u = s')}{D(r_u = k)} \cdot \frac{\sum_{v \in V_s} \frac{d(u, v)^{-s'-s}}{c(s')c(s)}}{\sum_{v \in V_s} \frac{d(u, v)^{-k-s}}{c(k)c(s)}}.$$

For a given node u and a subset of nodes Γ , let define $d_{\min, \Gamma}$ and $d_{\max, \Gamma}$ be the minimum and maximum grid distances between node u and any node $v \in \Gamma$, respectively. In other words, $d_{\min, \Gamma} \leq d_M(u, v) \leq d_{\max, \Gamma}, \forall v \in \Gamma$. With this definition, for any $v \in V_s$, we have:

$$\frac{d(u, v)^{-s'-s}}{c(s')c(s)} = \frac{c(k)}{c(s')} d(u, v)^{k-s'} \leq \begin{cases} \frac{c(k)}{c(s')} d_{\max, V_s}^{k-s'} & \text{if } s' < k, \\ \frac{c(k)}{c(s')} d_{\min, V_s}^{k-s'} & \text{if } s' > k. \end{cases}$$

Combing the above inequality, we have:

$$\frac{D(r_u = s') P_{u, V_s}(\mathbf{r})}{D(r_u = k) P_{u, V_s}(\mathbf{r})} \leq \begin{cases} \frac{D(r_u = s')/c(s')}{D(r_u = k)/c(k)} d_{\max, V_s}^{k-s'} & \text{if } s' < k, \\ \frac{D(r_u = s')/c(s')}{D(r_u = k)/c(k)} d_{\min, V_s}^{k-s'} & \text{if } s' > k. \end{cases} \quad (26)$$

We first show that $\pi(r_u = k, \mathbf{r}_{-u}) > \pi(r_u = s', \mathbf{r}_{-u})$ when $s' > k$. In the case of $s' > k$, as $d_{\min, V_s} \geq 1$, combining the above inequality with the bounds in Lemma 2 and Lemma 3, we get:

$$\frac{D(r_u = s') P_{u, V_s}(\mathbf{r})}{D(r_u = k) P_{u, V_s}(\mathbf{r})} = O\left(\frac{\ln^2(2kn)}{n^\gamma}\right). \quad (27)$$

Therefore, we can find a constant σ such that:

$$\begin{aligned} &\pi(r_u = k, \mathbf{r}_{-u}) - \pi(r_u = s', \mathbf{r}_{-u}) \\ &= \sum_{s \in S \cup \{0\}} [D(r_u = k) P_{u, V_s}(\mathbf{r}) - D(r_u = s') P_{u, V_s}(\mathbf{r})] \\ &= \sum_{s \in S \cup \{0\}} D(r_u = k) P_{u, V_s}(\mathbf{r}) \left(1 - \frac{D(r_u = s') P_{u, V_s}(\mathbf{r})}{D(r_u = k) P_{u, V_s}(\mathbf{r})} \right) \\ &\geq \sum_{s \in S \cup \{0\}} D(r_u = k) P_{u, V_s}(\mathbf{r}) \left(1 - \frac{\sigma \ln^2(2kn)}{n^\gamma} \right) > 0, \end{aligned} \quad (28)$$

for sufficiently large n .

We next show that $\pi(r_u = k, \mathbf{r}_{-u}) > \pi(r_u = s', \mathbf{r}_{-u})$ when $s' < k$. Note here we require the constant $\varepsilon < \gamma/2$ in the theorem. We first find a distance threshold to partition nodes into nodes nearby to u and nodes far away from u . We want to prove that $\pi(r_u = k, \mathbf{r}_{-u}) - \pi(r_u = s', \mathbf{r}_{-u})$ is dominated by the nearby nodes.

In the case of $s' < k$, we can find a constant $\nu = 1 - \frac{\gamma-2\varepsilon}{2k}$ such that, for any $s \in S$, the set V_s can be partitioned into two subsets: (i) $V_s^- = \{v \in V \mid r_v = s \wedge d_M(u, v) \leq n^\nu\}$, and (ii) $V_s^+ = \{v \in V \mid r_v = s \wedge d_M(u, v) > n^\nu\}$. Notice that d_{\max, V_s^-} is at most n^ν . Combing the above inequality Eq. (26) with the bounds in Lemma 2 and Lemma 3, we get:

$$\frac{D(r_u = s') P_{u, V_s^-}(\mathbf{r})}{D(r_u = k) P_{u, V_s^-}(\mathbf{r})} = O\left(\frac{\ln^2(2kn)}{n^{(1-\nu)\gamma}}\right), \quad (29)$$

where $\nu < 1$.

Notice that $P_{u,V_s} = P_{u,V_s^-} + P_{u,V_s^+}$. Based on the bound in Eq. (29), we can find a constant σ' such that:

$$\begin{aligned}
& \pi(r_u = k, \mathbf{r}_{-u}) - \pi(r_u = s', \mathbf{r}_{-u}) \\
& \geq \sum_{s \in S \cup \{0\}} D(r_u = k) P_{u,V_s^-}(\mathbf{r}) - \sum_{s \in S \cup \{0\}} D(r_u = s') P_{u,V_s}(\mathbf{r}) \\
& \geq \sum_{s \in S \cup \{0\}} D(r_u = k) P_{u,V_s^-}(\mathbf{r}) \left(1 - \frac{D(r_u = s') P_{u,V_s^-}(\mathbf{r})}{D(r_u = k) P_{u,V_s^-}(\mathbf{r})} \right) \\
& \quad - \sum_{s \in S \cup \{0\}} D(r_u = s') P_{u,V_s^+}(\mathbf{r}) \\
& \geq \sum_{s \in S \cup \{0\}} D(r_u = k) P_{u,V_s^-}(\mathbf{r}) \left(1 - \frac{\sigma' \ln^2(2kn)}{n^{(1-\nu)\gamma}} \right) \\
& \quad - \sum_{s \in S \cup \{0\}} D(r_u = s') P_{u,V_s^+}(\mathbf{r}) \\
& \geq \sum_{s \in S \cup \{0\}} \frac{D(r_u = k) P_{u,V_s^-}(\mathbf{r})}{2} - \sum_{s \in S \cup \{0\}} D(r_u = s') P_{u,V_s^+}(\mathbf{r})
\end{aligned} \tag{30}$$

for sufficiently large n .

We now give the lower bound of the first term $D(r_u = k) P_{u,V_s^-}(\mathbf{r})$. Let $U_j = \{v \mid d_M(u, v) = j \wedge r_v > 0\}$. By Lemma 6, with probability $1 - 1/n$,

$$|U_j| > \frac{\eta b_u(j)}{2}, \forall u \in V, \forall j \in \mathbb{N} \cap \left[\rho \left(\frac{\ln n}{\eta} \right)^{\frac{1}{k-1}}, \frac{n}{2} \right]. \tag{31}$$

For $j = \left\lceil \rho \left(\frac{\ln n}{\eta} \right)^{\frac{1}{k-1}} \right\rceil$, we have:

$$\begin{aligned}
P_{u,U_j}(\mathbf{r}) &= \sum_{v \in U_j} p_u(v, r_u) \cdot p_v(u, r_v) = \sum_{v \in U_j} \frac{j^{-k}}{c(k)} \cdot \frac{j^{-r_v}}{c(r_v)} \\
&\geq \sum_{v \in U_j} \frac{j^{-k}}{c(k)} \cdot \frac{j^{-\beta}}{c(\gamma)} \geq \frac{\eta \cdot b_u(j) \cdot j^{-k-\beta}}{2c(k)c(\gamma)} \\
&\geq \frac{\eta \cdot \xi_k^- j^{k-1} \cdot j^{-k-\beta}}{2c(k)c(\gamma)} \geq \frac{\eta \xi_k^- \rho^{\frac{-(\beta+1)}{k-1}} \left(\frac{\ln n}{\eta} \right)^{\frac{-(\beta+1)}{k-1}}}{2c(k)c(\gamma)}.
\end{aligned}$$

We now fix $\eta = 1/n^{\frac{(k-1)\varepsilon}{k+\beta}}$ ($0 < \varepsilon < \gamma/2$), and have:

$$P_{u,U_j}(\mathbf{r}) \geq \frac{\xi_k^-}{2\rho^{\frac{\beta+1}{k-1}} c(k)c(\gamma)} \cdot \frac{1}{(\ln n)^{\frac{(\beta+1)}{k-1}} n^\varepsilon}. \tag{32}$$

Note that $U_j \subseteq \cup_{s \in S \cup \{0\}} V_s^-$, since $j = \left\lceil \rho \left(\frac{\ln n}{\eta} \right)^{\frac{1}{k-1}} \right\rceil = \left\lceil \rho \ln^{\frac{1}{k-1}} n \cdot n^{\frac{\varepsilon}{k+\beta}} \right\rceil < n^{\frac{k-\gamma/2+\varepsilon}{k}} = n^\nu$ for sufficiently large n . Combing with the bounds in Lemma 2 and Lemma 3, we get:

$$\begin{aligned}
& D(r_u = k) \sum_{s \in S \cup \{0\}} P_{u,V_s^-} \\
& \geq D(r_u = k) P_{u,U_j}(r_u = k, \mathbf{r}_{-u}) = \Omega \left(\frac{n^{1-k-\varepsilon+\gamma}}{\ln^a(2kn)} \right),
\end{aligned} \tag{33}$$

where $a = 2 + \frac{\beta+1}{k-1}$ is constant.

We next give the upper bound of the second term $D(r_u = s') P_{u,V_s^+}$. Notice that $d_M(u, v) > n^\nu$ for any v in V_s^+ , so for

any s , we have:

$$P_{u,V_s^+}(\mathbf{r}) = \sum_{v \in V_s^+} \frac{d_M(u, v)^{-s'-s}}{c(s')c(s)} \leq \sum_{v \in V_s^+} \frac{n^{-\nu(s'+s)}}{c(s')c(s)}. \tag{34}$$

In the case of $s < k$, combining the above inequality with the bound in Lemma 2, we get:

$$\begin{aligned}
P_{u,V_s^+}(\mathbf{r}) &\leq \sum_{v \in V_s^+} \frac{2^{2k+2} k^2}{(\xi_k^-)^2} n^{(s+s')(1-\nu)-2k} \\
&\leq \sum_{v \in V_s^+} \frac{2^{2k+2} k^2}{(\xi_k^-)^2} n^{2k(1-\nu)-2k} = |V_s^+| \frac{2^{2k+2} k^2}{(\xi_k^-)^2} n^{-2k\nu}.
\end{aligned} \tag{35}$$

In the other case of $s \geq k$, combining the inequality Eq. (34) with the bounds in Lemma 2, we get:

$$\begin{aligned}
P_{u,V_s^+}(\mathbf{r}) &\leq \sum_{v \in V_s^+} \frac{n^{-\nu(s'+k)}}{c(s')\xi_k^-} \leq \sum_{v \in V_s^+} \frac{2^{k+1} k}{(\xi_k^-)^2} n^{(1-\nu)s'-(1+\nu)k} \\
&\leq \sum_{v \in V_s^+} \frac{2^{k+1} k}{(\xi_k^-)^2} n^{(1-\nu)k-(1+\nu)k} = |V_s^+| \frac{2^{k+1} k}{(\xi_k^-)^2} n^{-2k\nu}.
\end{aligned} \tag{36}$$

Combining the above inequalities Eq. (35) and Eq. (36) with the bound in Lemma 3, we know that for any s :

$$D(r_u = s') P_{u,V_s^+} = O(|V_s^+| n^{1-2k\nu}). \tag{37}$$

We are now ready to combine the above bounds and show that $\pi(r_u = k, \mathbf{r}_{-u}) > \pi(r_u = s', \mathbf{r}_{-u})$ when $s' < k$. More specifically, combining the inequalities in Eq. (30), Eq. (33) and Eq. (37), we get:

$$\begin{aligned}
& \pi(r_u = k, \mathbf{r}_{-u}) - \pi(r_u = s', \mathbf{r}_{-u}) \\
& \geq \sum_{s \in S \setminus \{0\}} \frac{D(r_u = k) P_{u,V_s^-}}{2} - \sum_{s \in S \cup \{0\}} D(r_u = s') P_{u,V_s^+}, \\
& \geq \frac{\rho n^{1-k+\gamma-\varepsilon}}{2 \ln^a(2kn)} - \rho' |\cup_{s \in S \cup \{0\}} V_s^+| \cdot n^{1-2k\nu} \\
& \geq \frac{\rho n^{1-k+\gamma-\varepsilon}}{2 \ln^a(2kn)} - \rho' n^k \cdot n^{1-2k+\gamma-2\varepsilon} \\
& \geq \frac{\rho n^{1-k+\gamma-\varepsilon}}{2 \ln^a(2kn)} - \rho' n^{1-k+\gamma-2\varepsilon},
\end{aligned} \tag{38}$$

where σ, ρ, ρ', a are all constants.

As $0 < \varepsilon < \gamma/2$, the first term in Eq. (38) is in strictly higher order in n than the second term in Eq. (38), we know that for sufficiently large n , $\pi(r_u = k, \mathbf{r}_{-u}) > \pi(r_u = s', \mathbf{r}_{-u})$ for any $s' < k$.

Therefore, when the property in Eq. (31) holds, the perturbed strategy profile \mathbf{r} moves to the navigable small world ($\mathbf{r}' \equiv k$) in one synchronous step, as every player u moves from its current strategy to its best response $r'_u = k$. Also, it is clear that the property Eq. (31) consistently holds as any player takes one asynchronous step. This is because the asynchronous move only increases the number of nodes choosing a non-zero strategy, so the best response of every player is always k after every asynchronous step. Thus, the perturbed strategy profile moves to the navigable small world as soon as every node takes at least one asynchronous step. Notice that the property in Eq. (31) holds with a probability of $1 - 1/n$, so the theorem is proved. \square

H. PROOF OF THEOREM 9

PROOF OF THEOREM 9. Given the strategy profile \mathbf{r} , we partition the nodes V into three sets: $V_{<k} = \{v \in V \mid r_v < k\}$, $V_{>k} = \{v \in V \mid r_v > k\}$, $V_{=k} = \{v \in V \mid r_v = k\}$. For any $A \subseteq V$, So we have:

$$\pi_u(\mathbf{r}) = D(r_u) (P_{u,V_{<k}}(\mathbf{r}) + P_{u,V_{>k}}(\mathbf{r}) + P_{u,V_{=k}}(\mathbf{r})). \quad (39)$$

For any node $u \in V_{=k}$, we have:

$$\begin{aligned} P_{u,V_{<k}}(\mathbf{r}) &= \sum_{v \in V_{<k}} \frac{d_M(u,v)^{-r_u-r_v}}{c(r_u)c(r_v)} \leq \sum_{v \in V_{<k}} \frac{d_M(u,v)^{-r_u}}{c(r_u)c(k-\gamma)} \\ &\leq \frac{\sum_{v \neq u} d_M(u,v)^{-r_u}}{c(r_u)c(k-\gamma)} = \frac{1}{c(k-\gamma)}. \end{aligned}$$

Combining the above inequality with bounds given by Lemma 2 and Lemma 3, we get the upper bound on the payoff obtained from the set $V_{<k}$:

$$D(r_u = k)P_{u,V_{<k}}(\mathbf{r}) \leq \frac{\xi_k^+ n}{c(k)c(k-\gamma)} \leq \frac{\xi_k^+ 2\gamma + 2\gamma}{(\xi_k^-)^2} \frac{n^{1-\gamma}}{\ln n}. \quad (40)$$

For the set $V_{>k}$, we have:

$$\begin{aligned} P_{u,V_{>k}}(\mathbf{r}) &= \sum_{v \in V_{>k}} \frac{d_M(u,v)^{-r_u-r_v}}{c(r_u)c(r_v)} \\ &\leq \frac{\sum_{j=1}^{n_D} b_u(j) \cdot j^{-r_u} \cdot j^{-k-\gamma}}{\xi_k^- c(r_u)} = \frac{\xi_k^+ \sum_{j=1}^{n_D} j^{-1-r_u-\gamma}}{\xi_k^- c(r_u)} \\ &\leq \frac{\xi_k^+ (1 + \int_1^{n_D} x^{-1-r_u-\gamma} dx)}{\xi_k^- c(r_u)} \leq \frac{\xi_k^+ (1 + r_u + \gamma)}{\xi_k^- (r_u + \gamma) c(r_u)} \leq \frac{\xi_k^+ (k+1)}{\xi_k^- \gamma c(r_u)}. \end{aligned}$$

Combining the above inequality with the bounds given by Lemma 2 and Lemma 3, we get the upper bound on the payoff obtained from the set $V_{>k}$:

$$D(r_u = k)P_{u,V_{>k}}(\mathbf{r}) \leq \frac{\xi_k^+ n}{c(k)} \frac{\xi_k^+ (k+1)}{\xi_k^- \gamma c(k)} \leq \frac{4(\xi_k^+)^2 (k+1)}{\gamma (\xi_k^-)^3} \frac{n}{\ln^2 n}. \quad (41)$$

For the set $V_{=k}$, by Lemma 5 we know that:

$$D(r_u = k)P_{u,V_{=k} \setminus \{u\}}(\mathbf{r}) \leq \pi(r_u = k, \mathbf{r}_{-u} \equiv k) \leq \frac{16(\xi_k^+)^2}{(\xi_k^-)^3} \cdot \frac{n}{\ln^3 n}$$

Then, for any node $u \in V_{=k}$ and sufficiently large n , we have

$$\begin{aligned} \pi_u(\mathbf{r}) &= D(r_u) (P_{u,V_{<k}}(\mathbf{r}) + P_{u,V_{>k}}(\mathbf{r}) + P_{u,V_{=k} \setminus \{u\}}(\mathbf{r})) \\ &\leq \frac{\xi_k^+ 2\gamma + 2\gamma}{(\xi_k^-)^2} \frac{n^{1-\gamma}}{\ln n} + \frac{4(\xi_k^+)^2 (k+1)}{\gamma (\xi_k^-)^3} \frac{n}{\ln^2 n} + \frac{16(\xi_k^+)^2}{(\xi_k^-)^3} \cdot \frac{n}{\ln^3 n} \\ &< \frac{8(\xi_k^+)^2 (k+1)}{\gamma (\xi_k^-)^3} \frac{n}{\ln^2 n}. \end{aligned} \quad (42)$$

By Lemma 4, for any node $u \notin V_{=k}$, there is a constant κ such that for all sufficiently large n , $\pi(r_u, \mathbf{r}_{-u}) \leq \kappa n^{1-\gamma}$.

So we have, for sufficiently large n , the social welfare of the profile is:

$$\begin{aligned} SW(\mathbf{r}) &= \sum_{u \in V} \pi(r_u, \mathbf{r}_{-u}) < |V| \left(\frac{8(\xi_k^+)^2 (k+1)}{\gamma (\xi_k^-)^3} \frac{n}{\ln^2 n} + \kappa n^{1-\gamma} \right) \\ &< \frac{16(\xi_k^+)^2 (k+1)}{\gamma (\xi_k^-)^3} \frac{n^{k+1}}{\ln^2 n}. \end{aligned} \quad (43)$$

The above inequality shows that the social welfare of the profile is at most $O\left(\frac{n^{k+1}}{\ln^2 n}\right)$.

Next, we construct the profile \mathbf{r} as follows: for any node with location (i, j) , we set its strategy as k if $j \bmod 2 = 0$, otherwise, we set its strategy as $k + \gamma$.

Notice that for any node u with $r_u = k$, it has at least one neighbor v with $r_v > k$. We get:

$$P_{u,V} > p_u(v, r_u)p_v(u, r_v) > \frac{1}{c(k)c(k+\gamma)}. \quad (44)$$

For any strategy $k + \gamma$, we have:

$$\begin{aligned} c(k + \gamma) &\leq \xi_k^+ \sum_{j=1}^{n_D} j^{-\gamma-1} \leq \xi_k^+ (1 + \int_1^{n_D+1} x^{-\gamma-1} dx) \\ &\leq \xi_k^+ \left(1 + \frac{1}{\gamma}\right). \end{aligned} \quad (45)$$

Combining with bounds in Lemma 2 and Lemma 3, for any node u with $r_u = k$,

$$\begin{aligned} \pi_u(\mathbf{r}) &> \frac{\xi_k^- n}{2c^2(k)c(k+\gamma)} > \frac{\xi_k^- \gamma}{2(\xi_k^+)^3(1+\gamma)} \frac{n}{\ln^2(2kn)} \\ &> \frac{\xi_k^- \gamma}{8(\xi_k^+)^3(1+\gamma)} \frac{n}{\ln^2 n}. \end{aligned} \quad (46)$$

So we have, for sufficiently large n , the social welfare of the profile is:

$$\begin{aligned} SW(\mathbf{r}) &= \sum_{u \in V} \pi(r_u, \mathbf{r}_{-u}) > \frac{|V|}{2} \frac{\xi_k^- \gamma}{8(\xi_k^+)^3(1+\gamma)} \frac{n}{\ln^2 n} \\ &> \frac{\xi_k^- \gamma}{16(\xi_k^+)^3(1+\gamma)} \frac{n^{k+1}}{\ln^2 n}. \end{aligned} \quad (47)$$

The above inequality shows that the optimal social welfare is at least $\Omega\left(\frac{n^{k+1}}{\ln^2 n}\right)$. Combining the results of Eq.(43) and Eq.(47), the theorem is proved. \square

I. PROOF OF THEOREM 10

PROOF OF THEOREM 10. According to Lemma 5 it is easy to get the social welfare of $\mathbf{r} \equiv k$ is $\Theta\left(\frac{n^{k+1}}{\ln^3 n}\right)$. Combining with Theorem 9, the price of stability (PoS) is $\Theta(\ln n)$.

For random small world $\mathbf{r}_{-u} \equiv 0$, we have:

$$P_u(r_u, \mathbf{r}_{-u} \equiv 0) = \sum_{v \in V} \frac{d_M(u,v)^{-r_u}}{c(r_u)c(0)} = \frac{1}{c(0)}. \quad (48)$$

It is easy to get:

$$c(0) = n^k - 1 \geq n^k/2. \quad (49)$$

Thus, combining the above inequality with the distance bound in Lemma 3, we have the payoff of node u gets from $V_{=0}$:

$$\pi(r_u, \mathbf{r}_{-u} \equiv 0) \leq \frac{2^{k+2}k^{2+k}\xi_k^+}{\xi_k^-} n^{1-k}. \quad (50)$$

According to the above inequality, it is easy to get that the social welfare of $\mathbf{r} \equiv 0$ is at most $O(n)$. We now examine

its lower bound. To do so, we first get the lower bound on distance.

$$\begin{aligned}
D(r_u = 0) &\geq \frac{\sum_{j=1}^{n/2} b_u(j) \cdot j}{c(0)} \geq \frac{\xi_k^- \int_1^{n/2} x^k dx}{c(0)} \\
&\geq \frac{\xi_k^- (n/2 - 1)^{1+k}}{(1+k)c(0)} > \frac{\xi_k^- (n/4)^{1+k}}{(1+k)c(0)}, \quad (51)
\end{aligned}$$

Combining the above inequality with Eq.(49) and Eq. (48), so we can get

$$\pi(r_u = k, \mathbf{r}_{-u} \equiv 0) > \frac{\xi_k^-}{4^{1+k}} n^{1-k}. \quad (52)$$

Therefore, the social welfare of the random small-world network ($\mathbf{r} \equiv k$) is $\Theta(n)$. Combining with Theorem 9, the price of anarchy (PoA) is $\Theta\left(\frac{n^k}{\ln^2 n}\right)$. \square